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DEVELOPING MATHEMATICS EDUCATION IN A SYSTEMIC PROCESS *

The aim of this paper is to make a concrete proposal for bridging the gap between theory and practice in mathematics education and for establishing a systemic relationship between researchers and teachers as well as to explain the background and the implications of this proposal.

1. BRIDGING THE GAP BETWEEN THEORY AND PRACTICE: THE ROLE OF SUBSTANTIAL LEARNING ENVIRONMENTS

> You cannot fail if you follow the advice the genius of human reason whispers in the ear of each new-born child, namely to test thinking by doing and doing by thinking.
> J.W. von Goethe

In Guy Brousseau's book 'Theory of Didactical Situations in Mathematics' the scene is set with a teaching example, the 'race to 20', which is based on a game of strategy (Brousseau 1997, 3–18). In a somewhat modified version this game can be described as follows (cf. Figure 1):

Figure 1.

A line of circles is numbered from 1 to 20. The first player starts by putting 1 or 2 counters on the first circle or the first two circles, the second player follows by putting 1 or 2 counters on the next circles similarly.

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Continuing in this way the players take turns until one of them arrives at 20 and in doing so wins the game.

The ‘race to 20’ helps to corroborate basic arithmetical ideas (relationships of numbers on the number line, addition, repeated addition). It is also a rich context for general objectives of mathematics education (exploring, reasoning and communicating) and a typical example of the fundamental principle of ‘learning by inquiry’. If children analyse the moves backwards they recognise that the positions 17, 14, 11, 8, 5 and 2 are winning positions. So the first player has a winning strategy: In the first move she puts down two counters and then responds to a 2-counters move of the second player with a 1-counter move and to a 1-counter move of her opponent with a 2-counters move. In this way the first player jumps from one winning position to the next one and finally arrives at 20.

There are many variations of this game: Any natural number can be chosen as the target, and the maximal number of counters to be put down on every move can be increased. In fact we have a whole class of games of strategy before us which require a continuous adaptation of the strategies used.

The basic ideas of analysing these games can be generalised to the wider class of finite deterministic games of strategy for two persons with full information which cannot end in a draw: by means of the game tree and the marking algorithm one can prove that for each of these games there exists a winning strategy either for the first or the second player.

As mentioned in Brousseau’s book the ‘race to 20’ was reproduced 60(!) times under observation and each of its phases was the object of experimentation and clinical study. Based on a variety of other teaching examples Brousseau developed his theory of didactical situations. In the research context ‘aspects of proving’ Galbraith (1981) studied students’ psychological processes in their attempts to uncover the structure underlying the ‘race to 25’.

The ‘race to 20’ and its variations represent what has been called a substantial learning environment, an SLE, that is a teaching/learning unit with the following properties (Wittmann, 1995, 365/366):

(1) It represents central objectives, contents and principles of teaching mathematics at a certain level.
(2) It is related to significant mathematical contents, processes and procedures *beyond* this level, and is a rich source of mathematical activities.
(3) It is flexible and can be adapted to the special conditions of a classroom.
(4) It integrates mathematical, psychological and pedagogical aspects of teaching mathematics, and so it forms a rich field for empirical research.

The concept of an SLE is a very powerful one. It can be used to tackle successfully one of the big issues of mathematics education which has become more and more urgent and which is of crucial importance for the future of mathematics education as a discipline: the issue of theory and practice. Fortunately, for some years now this issue has been more and more recognised and addressed by mathematics educators. Referring to the recent ICMI Study 'Mathematics Education as a Research Domain' (Sierpinska and Kilpatrick 1998) Ruthven stated that there is a wide gap between the scholarly knowledge of researchers on the one hand and the craft knowledge of teachers on the other hand, and argued in favour of a re-orientation of mathematics education:

> "While most of the contributors identify the development of knowledge and resources capable of supporting the teaching and learning of mathematics as an important goal for the field, there is disappointment with what has been demonstrated on this score." (Ruthven, 2001)

A claim similar to Ruthven's was made by Clements and Ellerton for the South East Asian and by Stigler and Hiebert for the American context:

> "From our perspective, at the present time mathematics education needs less theory-driven research, and more reflective, more culture-sensitive, and more practice-orientated research which will assist in the generation of more domain-specific theory." (Clements and Ellerton, 1996, 184)

> "Perhaps what teachers are told by researchers to do makes little sense in the context of an actual classroom. Researchers might be very smart. But they do not have access to the same information that teachers have as they confront real students in the context of real lessons with real learning goals . . . It is clear that we need a research-and-development system for the steady, continuous improvement; such a system does not exist today." (Stigler and Hiebert, 1999, 126–127)

This criticism can be extended: for teachers' decision-making the logical and epistemological structure of the subject matter is at least as important as are psychological, social or more general aspects of learning and teaching. However, in the mainstream of current research in mathematics education this very structure has not received the attention it deserves. Therefore the gap between theory and practice is also due to a gap between mathematics on one side and mathematics education on the other side. This gap is particularly obstructive to progress in reforming mathematical education as the epistemological structure of the subject matter contains psychological and social aspects at least implicitly while the converse does not hold.

Of course the issue of relating theory and practice to one another is not specific for mathematics education. In all fields we have, at one extreme, mere 'doers' who act in a pragmatic manner, who don't see any point in worrying about theory and who even think of theory as a threat to practice. At the other extreme are mere ‘thinkers’ who develop analyses and theories with no grounding in practice and without caring for practical implications and applications.

In tackling the issue of theory and practice a superficial re-arrangement of the field is not sufficient. If we seriously want to establish links between theory and practice a fundamental change is needed. For systemic reasons it is highly unlikely that theories which have been developed independently of practice can be applied afterwards:

> “The developing theory of mathematical learning and teaching must be a refinement, an extension and a deepening of practitioner knowledge, not a separate growth”

as stated by Alan Bell in the mid-eighties (Bell 1984, 109).

Therefore in order to organise a strong and lasting systemic interaction between theorists and practitioners we have to look for some common core in which theory and practice as well as mathematics and mathematics education inseparably permeate one another. Substantial learning environments can serve this purpose quite naturally (cf., Wittmann, 1984, and Wittmann, 1995/1998). Accordingly, the main proposal of this paper is as follows (see Figure 2 which is an extension of a diagram presented in Ruthven 2000):

The design of substantial learning environments around long-term curricular strands should be placed at the very centre of mathematics education. Research, development and teacher education should be consciously related to them in a systematic way.

This proposal is supported by encouraging experiences which have been made in various projects around the world. Prominent examples are the work of the British Association of Teachers of Mathematics in the sixties (cf., Fletcher, 1965; Wheeler, 1967), the prolific Dutch *Wiskobas* project and its follow-up projects conducted at the Freudenthal Institute, and the systematic work of Japanese mathematics educators (cf., Shimada and Becker, 1996). These projects show what an important role SLEs can play for both researchers and practitioners: as common points of reference, as knots in the collective memory, and as stimuli for action. The proposal reflects a certain understanding of the particular nature of the system of education which will be examined in the next section.

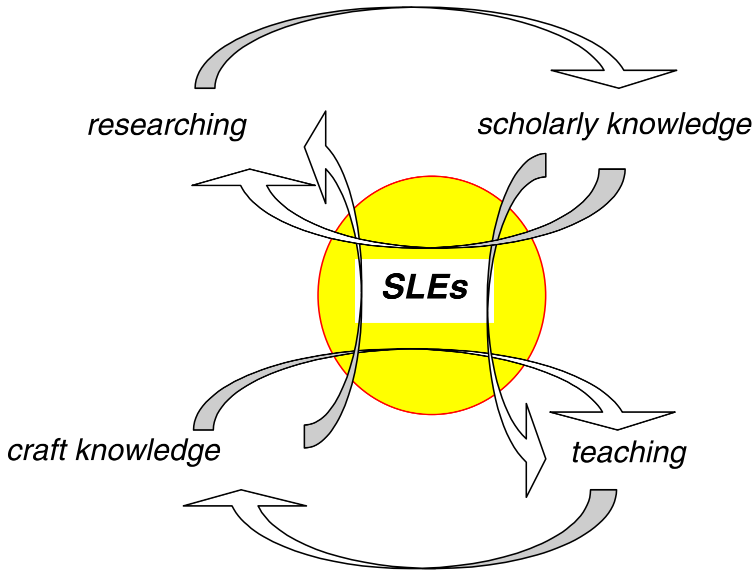


Figure 2.

2. (BURST) DREAMS

Variety can only be absorbed by variety.
Ross Ashby

It is not by chance that development projects based on SLEs have been successful in changing mathematics teaching as well as in changing teachers' attitudes: in these projects fundamental systemic conditions have been taken into account. This will be explained in more detail by referring to three dreams that were dreamt by a prominent philosopher, a prominent mathematician and a prominent educator. These dreams have been selected because they capture the non-systemic tradition of teaching and learning which must be fully recognised in order to be overcome.

2.1. *Descartes' dream*

In 1619 the young René Descartes (1596–1650) had a vision of the 'foundations of a marvellous science', on which the elaborated later in several writings, particularly in his *'Discourse on the method of properly guiding the reason in the search of truth in the sciences'* (Descartes, 1637). Basically this method consisted of a few rules by which the mind can arrive at more and more complete descriptions of reality. In modern words the method was a totalitarian programme for mathematising reality. By separating the thinking mind, the *res cogitans*, from the world outside, the *res extensa*, Descartes established a sharp split between man and his environment which later on became a fundamental ideology of Western thinking. Already before Descartes Francis Bacon (1561–1626) had formulated the inductive method of science and summarised its technological use in the slogan 'Knowledge is Power'. So from the very beginning Descartes' dream of arriving at a complete description of the environment was accompanied by the dream of controlling and making use of it. The 'Cartesian system' of philosophy, as it was called later, has paved the way for an unrestrained mathematisation, control and also exploitation of more and more parts of our natural and social environment. In our time the availability of computers has accelerated this process (cf. Davis and Hersh, 1988). 'Benchmarking', 'controlling', 'evaluation', and 'assessment' have become key notions in the management hierarchies of economics and administration.

2.2. *Hilbert's dream*

At the turn to the 20^{th} century the very science in which Descartes wanted to ground truth, mathematics, was fundamentally shaken by the discovery of inconsistencies within Cantor's set theory. Among those mathematicians who were particularly alarmed was David Hilbert. In order to defend 'the paradise', which, in his eyes, Cantor had created, he started the so called 'finitistic programme' by which he hoped to prove the consistency and infallibility of mathematical theories once and for all (Hilbert, 1926). Although Hilbert's dream burst already in 1930 when Gödel proved his incompleteness theorem, the formalistic setting of Hilbert's programme has survived and turned into an implicit theory of teaching and learning. Interestingly, the Bourbaki movement which set formal standards in mathematics up to the seventies started in the mid-thirties from a discussion about how to teach analysis. Also in this day and age the belief in formal precision as a necessary if not sufficient means to manage teaching/learning processes is still widespread among mathematicians and non-mathematicians. The *Mathematically Correct* movement, at present one of the most aggressive pressure groups in the U.S., is a horrifying example.

2.3. *Comenius' dream*

Johann Amos Comenius (1592–1670) is well-known as one of the founding fathers of didactics. His famous book 'Great Didactic' published in 1657 was the first comprehensive work on teaching and learning. In many respects Comenius was far ahead of his time. For example, he was among the first to project a plan of universal education and to see the significance of education as an agency of international understanding. In one respect, however, he was a child of his time. Deeply impressed by Bacon's visions of a technological age and by the efficiency of newly invented machines, he was obsessed by the idea of transposing the functioning of machines to the functioning of teaching. In the chapters 13 and 32 of the 'Great Didactic' he states:

> The art of teaching, therefore, demands nothing more than the skilful arrangement of time, of the subjects taught, and of method. As soon as we have succeeded in finding the proper method it will be no harder to teach school-boys, in any number desired, than with the help of the printing-press to cover a thousand sheets daily with the neatest writing . . . The whole process will be as free from friction as is the movement of a clock whose motive power is supplied by the weights. It will be as pleasant to see education carried out on my plan as to look at an automatic machine of this kind, and the process will be as free from failure as are these mechanical contrivances, when skilfully made . . . Knowledge can be impressed on the mind, in the same way that its concrete form can be printed on paper. (Comenius, 1910, 96–97, 289)

Comenius' dream has been dreamt over the centuries in ever new forms and is still present in some corners of cognitive science and education, including mathematics education, as various forms of 'direct teaching' and 'hard science'-like methods of research demonstrate (cf., for example, Begle, 1979).

Also a certain tendency within the research community to consider teachers as mere recipients of research results is clearly related to Comenius' dream:

> "I suspect that if teachers are mainly channels of reception and transmission, the conclusions of research will be badly deflected and distorted before they get into the mind of pupils. I am inclined to believe that this state of affairs is a chief cause for the tendency . . . to convert scientific findings into recipes to be followed. The human desire to be an 'authority' and to control the action of others does not, alas, disappear when a man becomes a researcher.' (Dewey, 1929/1988, 24)

2.4. *The 'systemic-evolutionary' vs. the 'mechanistic-technomorph' approach to the management of complexity*

It may seem as too far-fetched to look at Descartes, Hilbert and Comenius from the point of view of modern systems and management theory. However, there is a good reason to do so, for the three dreams, as different as they may appear, share a common feature: They reflect the self-concept of individuals who perceive themselves as standing on a higher level and as equipped with the capacity to gather complete information about some field and to use this information for bringing this field under control. The Swiss management theorist Malik has called this attitude the 'mechanistic-technomorph approach to the management of complexity' and described it as follows:

> The paradigm [underlying this approach] is the machine in the sense of classical mechanics. Basically, a machine is constructed according to a given purpose and to a given plan, and its function, reliability and efficiency depend on the functions and the properties of its elementary components ... The technological success which has been achieved by following this paradigm is overwhelming, and gave rise to the belief in its unlimited applicability far beyond the engineering disciplines. ... The paradigm includes the firm conviction that no order whatsoever which corresponds to human purposes can be brought about without following this paradigm. (Malik, 1986, 36 ff., transl. E.Ch.W.)

During the past decades another paradigm has been taking ground, based on the fact that biological and social organisms are far too complex in order to allow for a 'mechanistic-technomorph' description and control from outside. In order to achieve certain goals with living systems a fundamentally different approach is appropriate:

> The systemic-evolutionary approach [to the management of complexity] starts from quite different assumptions. Its basic paradigm is the spontaneous, self-generating ordering exemplified best by the living organism. Organisms are not constructed, they develop. Spontaneous orderings develop also in the social domain. They arise by means of and as the result of human actions, but they do not necessarily correspond to preconceived intentions, plans or goals. Nevertheless they can be highly rational. (Malik, 1986, 38 ff, transl. E.Ch.W.)

According to the systemic-evolutionary paradigm the only reasonable and feasible way of influencing and guiding a social system is to interact sensibly with the self-organising powers inside the system. Recommendations and instructions from outside which do not fit into the internal processes of the system are, at best, useless. If, in addition, a minute control is exerted from the outside, the development of spontaneous powers inside the system is suppressed, and this undermines its efficiency. A system without a proper infrastructure is not able to interact adequately with a complex environment: variety can only be absorbed by variety.

The systemic-evolutionary approach to the management of complexity has been developing in Western philosophy only during the last decades. So it is even more astounding that it has emerged in Asia more than 2000 years earlier when Lao Tzu and Chuang Tzu founded the philosophy of taoism. The basic maxim of taoism for leaders is 'wu wei'. This means: leaders should not interfere with the natural powers and inclinations of their clients, but should instead build upon self-organisation and offer help for self-help. It is the present author's hope that the Asian societies will succeed in preserving the systemic-evolutionary thinking as a precious heritage from their past while it is spreading only slowly and with great difficulties in Western societies which are still in the claws of deeply rooted mechanistic-technomorph patterns of thinking and action.

3. Consequences for Mathematics Education

A little child needs no famous teacher to learn to speak. He or she learns to speak spontaneously in the company of people who can speak.
Chuang Tzu

Individual students, individual teachers, classrooms, staffs, school districts, states, countries: all are living organisms and therefore highly complex systems. *Beyond any political or educational ideologies* the following systemic conclusions can be drawn just from the natural law of the inherent complexity of these systems:

1. Learning unfolds best if the spontaneous powers of all involved are brought to bear and encouraged, and if autonomy and self-responsibility are developed.

The inevitable results of – possibly well-intended – straitjacket schemes of teaching, assessment and accountability are "over-standardisation, over-simplification, over-reliance on statistics, student boredom, increased numbers of dropouts, a sacrifice of personal understanding and, probably, a diminution of diversity in intellectual development." (Stake, 1995, 213).

2. The traditional borderline between the researcher on one side and the teacher on the other side has to be abandoned. Research has to build upon the spontaneous powers of teachers in the same way as teaching has to build upon the spontaneous powers of students.

Donald Schön described this new relationship between theorists and practitioners very convincingly in his book *The Reflective Practitioner* (Schön 1983, 323):

"In the kinds of reflective research I have outlined, researchers and practitioners enter into modes of collaboration very different from the forms of exchange envisaged under the model of applied science. The practitioner does not function here as a mere user of the researcher's product. He reveals to the reflective researcher the ways of thinking that he brings to his practice, and draws on reflective research as an aid to his own reflection-in-action."

3. At all levels the traditional hierarchies have to be transformed into networks of co-operation and mutual support.

A good account of what this means in different contexts is given in Burton 1999.

Although the ‘mechanistic-technomorph’ paradigm of management is still dominant in all fields of society around the world, the awareness of the systemic nature of teaching and learning is steadily growing. As far as research and development in mathematics education are concerned there are already innovative research programmes which follow the new paradigm with remarkable success, for example *developmental research* (Freudenthal, 1991; Gravemeijer, 1994), Guy Brousseau’s *theory of didactical situations* (Brousseau, 1997), the Japanese *lesson studies* (Stigler and Hiebert, 1999, chapter 7), and *action research* (cf. Ahmed and Williams, 1992; Clements and Ellerton, 1996, chapter 5).

The impact of these research programmes on practice rests on the fact that they are systematically focused on the design and empirical research of SLEs. A firm basis for a systemic researcher-teacher-interaction for ‘SLE studies’ is thus provided, as illustrated by the following examples.

Example 1

In German primary schools the traditional approach to arithmetic in grade 1 has been to introduce the number space 1 to 20 step by step: The first quarter of the school year is restricted to the numbers 1 to 6, the second quarter to the numbers 1 to 10. The third quarter is open to numbers 1 to 20, however, tasks like 7+5, in which the 10 has to be bridged, are postponed to the last quarter of the school year. Moreover, children are expected to follow the arithmetic procedures given by the teacher.

In the project ‘mathe 2000’ this traditional approach was challenged and replaced by a holistic approach: The open number space 1 to 20 is introduced fairly quickly as one whole, children are encouraged to start from their own strategies and are not restricted to just one procedure. This new approach was formulated and published as a connected series of SLEs in a handbook for practising skills (Wittmann and Müller, 1990; Grade 1: chapters 1–3). It was based on a systematic epistemological analysis of arithmetic, on inspirations from the developmental research conducted at the Freudenthal Institute (cf., Treffers et al., 1989/1990; van den Heuvel-Panhuizen, 1996) and on the intuitions of the designers. It was not based on empirical research conducted by professional researchers. Empirical studies, which confirmed the holistic approach, came only later (cf. Selter, 1995; Hengartner, 1999). Thus teachers were the first to try it out in their practice and they found that it works better than the traditional approach. Through the existing networks of teachers this new approach has spread widely in a remarkably short period of time. An innovative textbook is presently available (Wittmann and Müller, 2000), based on the holistic approach, and its wide acceptance by teachers has convinced authors of traditional textbooks to modify their approach.

From the systemic point of view the success of this innovation is not surprising:

> One might ask the general question whether, in the present state of our knowledge about mathematical education, we should progress faster by collecting 'hard' data on small questions, or 'soft' data on major questions. It seems to me that only results related to fairly important practitioner questions are likely to become part of an intelligent scheme of knowledge . . . Specific results unrelated to major themes do not become part of communal knowledge. On the other hand, 'soft' results on major themes, if they seem interesting and provocative to practitioners, get tested in the myriad of tiny experiments which teachers perform every day when they 'try something and see if it works'. (Bell, 1984, 109)

Example 2

The second example is an SLE from the Japanese 'open-ended approach'. This unit was thoroughly researched before it was published (Hashimoto, 1986; Hashimoto and Becker, 1999). Its guiding problem is the so-called 'matchstick problem': Children are shown a linear arrangement of squares (Figure 3) and asked to find out how many matchsticks are needed to build 5, 6, 7 or more squares.

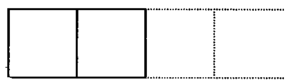


Figure 3.

There is a great variety of counting strategies to solve this problem. After having discussed the various solutions, children determine the number of matchsticks needed for other numbers of square, and try to find a general formula. In a similar way arrangements of matchsticks with more rows can be studied. Based on these concrete examples fundamental counting principles of combinatorics can be extracted, for example the addition principle and the principle of multiple counting (cf., Schrage, 1994).

Systematic lesson studies of the matchstick problem provided exactly the professional knowledge teachers need in order to teach this unit successfully. The matchstick problem has then been included in a textbook (Seki et al., 1997, 117–118).

Stigler/Hiebert comment on the impact of lesson studies as follows:

> The knowledge contained in these reports . . . is not made up of principles devoid of specific examples or examples without principles. It is theories linked with

examples. This knowledge is notable in several respects. First, theoretical insights are always linked with specific referents in the classroom. When a lesson-study group reports, for example, that one of its hypotheses has been supported, it is never outside the context of a specific lesson with specific goals, materials, students, and so on, all of which would be described in the report.

(Stigler and Hiebert, 1999, 163)

Example 3

A third example is provided by Heinz Steinbring's empirical research on the interplay between the epistemological structure of the subject matter and psychological and social factors (cf., for example, Steinbring, 1997). Although highly theoretical, his research is strongly related to SLEs which are part of the current teaching practice. So the applicability of the research results is guaranteed from the very outset.

4. Substantial learning environments for practising skills

What counts is not memorising, but understanding, not watching, but searching, not receiving, but seizing, not learning, but practising.
A. Diesterweg

Focusing mathematics education on substantial learning environments involves the risk that must be clearly recognised in order to be avoided: substantial mathematics is fundamentally related to mathematical processes such as mathematising, exploring, reasoning and communicating. These are *higher order thinking skills*. Emphasising them can easily lead to neglecting *basic skills*, in particular at a time when efficient calculators and computers are available. Basic skills also tend to be neglected for another reason: to a large extent traditional ways of teaching mathematics consisted of prescribed procedures and their stereotyped practice. In their eagerness to get rid of 'teach them and drill them' routines in favour of 'constructivist' ways of learning and teaching reformers easily get trapped: they tend to identify practice with stereotyped practice, and by abolishing stereotypes they are likely to do away with the practice of skills at all.

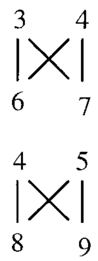
As the mastery of basic skills is an indispensable element of mathematical competence we have to find ways how to integrate the practice of skills into substantial mathematical activities. This is not an easy task, as stated, for example, by Ken Ross:

... drills of important algorithms that enable students to master a topic, while at the same time learning the reasoning behind them, can be used to great advantage by a knowledgeable teacher. Creative examples that probe students' understanding are difficult to develop but are essential. (Ross, 1998, p. 253)

The following example of an SLE (cf. Wittmann and Müller, 1990, grade 2, chapter 3.3) illustrates how the practice of a basic skills and the development of higher order skills can be combined. The example refers to an area of arithmetic which is notorious for drill and practice: the multiplication table.

The epistemological structure of the unit is unfolded in a heuristic manner as this is the best way to capture the potential of an SLE for both teaching and research (see also section 5.1).

The rule on which the unit is based is very simple: With two arbitrarily chosen pairs of consecutive numbers two calculations are performed: one 'top down', the other one 'crosswise' (Figure 4).



$$3 \cdot 6 + 4 \cdot 7 = 18 + 28 = 46$$

$$3 \cdot 7 + 4 \cdot 6 = 21 + 24 = 45$$

$$4 \cdot 8 + 5 \cdot 9 = 32 + 45 = 77$$

$$4 \cdot 9 + 5 \cdot 8 = 36 + 40 = 76$$

Figure 4.

After sufficiently many calculations with numbers chosen by the children themselves a pattern is recognised: The result obtained 'top down' seems always 1 bigger than the result obtained 'crosswise'. Children who have found pairs for which this relationship does not hold will spot some mistake in their calculations.

In trying to explain the pattern children have to go back to the meaning of multiplication: 3·6 means 6+6+6, 4·7 means 7+7+7+7, etc. So 3·6 + 4·7 contains one 7 more and one 6 less than 3·7 + 4·6 which gives it an advantage of 1 (Figure 5).

$$3 \cdot 6 + 4 \cdot 7 = 6 + 6 + 6 + 7 + 7 + 7 \ + 7 = 18 + 21 + 7 = 46$$

$$3 \cdot 7 + 4 \cdot 6 = 7 + 7 + 7 + 6 + 6 + 6 \ + 6 = 21 + 18 + 6 = 45$$

Figure 5.

Of course the standard proof of this relationship is an algebraic one employing variables which are not available in grade 2. But variables are not needed at this level, the argument used above is absolutely appropriate. As a next step the distributive law can be made more explicit. For example, $3 \cdot 6 + 4 \cdot 7$ can be written as $3 \cdot 6 + 3 \cdot 7 + 7$ and compared with $3 \cdot 7 + 4 \cdot 6$ written as $3 \cdot 7 + 3 \cdot 6 + 6$. This pre-algebraic form is an excellent preparation for algebra in higher grades. As is typical for substantial learning environments the activity can be extended: Instead of pairs of consecutive numbers pairs of numbers which differ by 2, 3 or any other number can be chosen. In Figure 6 the differences are 2.

3 5	$3 \cdot 6 + 5 \cdot 8 = 18 + 40 = 58$
6 8	$3 \cdot 8 + 5 \cdot 6 = 24 + 30 = 54$
7 9	$7 \cdot 4 + 9 \cdot 6 = 28 + 54 = 82$
4 6	$7 \cdot 6 + 9 \cdot 4 = 42 + 36 = 78$

Figure 6.

The differences can also be mixed (Figure7).

3 5	$3 \cdot 4 + 5 \cdot 7 = 12 + 35 = 47$
4 7	$3 \cdot 7 + 5 \cdot 4 = 21 + 20 = 41$
6 8	$6 \cdot 7 + 8 \cdot 10 = 42 + 80 = 122$
7 10	$6 \cdot 10 + 8 \cdot 7 = 60 + 56 = 116$

Figure 7.

From these examples a general pattern is emerging: the difference of the results of the two calculations is the product of the differences of the given numbers. It is not difficult to generalise the above proof for the introductory case. All one has to do is to decompose the second product in both calculations according to the distributive law.

Furthermore: beyond pairs of numbers triples of consecutive numbers (Figure 8) and triples with fixed differences can be considered. In this case the ‘top down’ result can be compared with two other results: one obtained by multiplying cyclically ‘from left to right’, the third one obtained by multiplying cyclically from ‘right to left’. In this case each triple

$$\begin{array}{ccc}3&4&5\\ |&|&|\\ 6&7&8\end{array} \qquad 3\cdot 6+4\cdot 7+5\cdot 8 = 18+28+40 = 86$$

$$\begin{array}{ccccc}3&&4&&5\\ &\diagdown&&\diagdown&\\ 6&&7&&8\end{array} \qquad 3\cdot 7+4\cdot 8+5\cdot 6 = 21+32+30 = 83$$

$$\begin{array}{ccccc}3&&4&&5\\ &\diagup&&\diagup&\\ 6&&7&&8\end{array} \qquad 3\cdot 8+4\cdot 6+5\cdot 7 = 24+24+35 = 83$$

Figure 8.

involves nine multiplications. Of course also triples with higher differences can be studied, and triples with different differences can be mixed as well. Furthermore: Pairs and triples can be generalised to n-tuples. Also more advanced mathematics can be employed as the expressions are scalar products of vectors . . .

In grade 2 or 3 only a tiny section of this very substantial learning environment can be explored. However, this does not reduce its importance for mathematics education as will be shown in the final section.

5. Substantial learning environments in teacher education

> I never force a piece of wood into a salad bowl.
> It's a raw material, living and talking.
> P. Peeters, Belgian wood artist

Efforts to establish a systemic relationship between theory and practice must include teacher education as it is in this field that the foundations for being able to act as a reflective practitioner are laid. SLEs, if properly used, can play a fundamental role here, too. It is appropriate to discuss didactical and mathematical courses separately as their positions in teacher education are different.

5.1. *Didactics courses*

The use of substantial learning environments is obvious for the didactic education of student teachers, that is for methods courses. By their very design SLEs offer unique possibilities for linking theoretical principles to concrete examples. If student teachers leave university with the intimate knowledge of theory-based learning environments they have at their disposal a professional background that will help them immensely to act as reflective practitioners. As convincingly explained in chapter 6 of Stigler and Hiebert (1999, p. 85 ff.) teaching is a cultural activity that can only be understood by becoming active in this culture. For this reason the best way for student teachers to capture the spirit of a substantial learning environment is to explore its epistemological structure, to reflect on it in terms of didactic principles, and to test their anticipations in the light of practical experiences. John Dewey gave a wonderful account of this 'laboratory point of view' in his fundamental paper *The Relation of Theory and Practice in Education* first published almost 100 years ago (Dewey, 1976).

In the last few years enormous progress has been made in applying the new technological possibilities of Multimedia to teacher education (cf. Lampert and Ball, 1998). Here SLEs can be of great help in order to identify teaching episodes that are substantial, mathematically and didactically, theoretically and practically, and to establish a well-structured and manageable information system that reflects the contents, objectives and principles of teaching mathematics at the corresponding level.

5.2. *Mathematics courses*

It is a simple matter of fact that around the world mathematical courses or even whole programmes often make only little or no sense for student teachers, for various reasons. Either the relevant subject matter is not covered at all, or the mathematical substance is stifled by a formalistic style of presentation or, even worse, there *is no substance*: mathematics is reduced to conceptual or procedural skeletons. Nevertheless, it would be wrong to conclude from meaningless courses that mathematical courses proper are of no use, in principle, for student teachers, and that the necessary mathematics should better be integrated into courses in mathematics education. On the contrary, a specific understanding of subject matter is of paramount importance for teachers as was convincingly explained, for example, by John Dewey in the paper mentioned above (Dewey, 1976). Dewey's arguments are based on a genetic perspective. He saw scientific enquiry as a social process and knowledge as a result of it.

From this perspective Dewey's emphasis on teachers' subject matter knowledge must not be taken as an unconditional support for mathematical courses of any kind but for courses which meet specific criteria. Courses in the context of specialised mathematics are perhaps appropriate for prospective mathematicians in industry or in research. However, from the point of view of mathematics education, it is counterproductive to take such courses as a model for teacher education. To consider specialised mathematics as something absolute and as a yardstick for the mathematical training in any other professional context would be a fundamental mistake.

It is a well established fact in the psychology of learning that knowledge cannot be acquired as a formal structure independently of the context in which it is to be used. Therefore, what is needed for teacher education is an idea of *mathematics in the educational context,* as formulated, for example, by Freudenthal:

> The idea of transposing academic mathematics (savoir savant) down to school mathematics (savoir enseigné) is wrong at its very outset, because the thinking behind this idea is directed top down and not bottom up. The mathematics to be learned at school by the big majority of our prospective citizens does not correspond at all to any theories of academic mathematics from which it could be watered down (didactically or not); at best it corresponds to the mathematics of scholars who lived centuries ago. The vast majority of our young people must be prepared to a technological know how (at various levels), not to the special knowledge of experts. The role of academic mathematics within this technological culture is much more modest than it has been claimed since a quarter of a century.. . .. (Freudenthal, 1986, transl. E.Ch.W.)

To postulate a specific conception of mathematics in the educational context has implications for both contents and methods. Elementary topics which are closely related to the curriculum are far more important for teachers than advanced topics, and above all student teachers must experience mathematics as an *activity* (Freudenthal, 1973; Wittmann, 2001). It is only in this way that they can learn to deal with elementary mathematical structures in a productive way and to play their role as reflective practitioners also with respect to contents: SLE studies presuppose a flexible mastery of the content.

In order to make mathematical courses meaningful for teacher education they should be systematically related to SLEs. By their very definition SLEs are based on substantial mathematics *beyond* the school level. Therefore every SLE offers mathematical activities for student teachers on a higher level. However, disconnected pieces of mathematical islands attached to scattered SLEs do not serve the purpose. What is needed in teacher education are systematic and coherent courses of elementary mathematics which cover the mathematical background of a variety of SLEs. To develop such courses is a challenging problem for the next decade. Within the project 'mathe 2000' a special series *Mathematics as a Process* has been started which is an attempt in this direction (cf., Müller/Steinbring/Wittmann, 2001).

Focusing the mathematical education of student teachers on substantial learning environments can also serve another purpose. At a time when education in general is in danger of being subordinated to economic purposes and to methods of mass production, when, as a consequence, mathematics at school is in danger of being reduced to a toolkit for applications, and teaching is in danger of preparing students just for passing tests, the following point is crucially important: Student teachers of all levels must experience the aesthetics of a genuine mathematical activity leading to the creation of structural wholes. The presence of SLEs in the mathematical studies can contribute to making student teachers aware that mathematics is not a homogeneous mass that can be cut into arbitrary pieces and forced into instructional schemes. Mathematical structures are living organisms, and learning processes must follow their inherent dynamics if learning mathematics is to make a deeper sense.

CONCLUSION

The systemic approach to the management of complexity on which this paper is based is more than just a scientific paradigm: basically, it is a way of life carried by an enlightened self-interest and directed towards sustainable development and co-existence. Therefore it is appropriate to conclude the paper with the systemic postulates for a future society formulated by Heinz von Foerster, the great master of systemic thinking (von Foerster, 1984):

1. Education is neither a right nor a privilege: it is a necessity.
2. Education is learning to ask questions to which the answers are not known.
3. A is better off if B is better off.

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Part of developing a social perspective means looking at the opportunities for numeracy in other curriculum areas. All too often this is interpreted as numeracy travelling out into other curriculum areas, but Sullivan raises the important issue of making opportunities within the mathematics lesson to explore other aspects of the curriculum.Â Mike Askew, formerly Professor of Mathematics Education at Kingâ€™s College London, is now Professor of Primary Education at Monash. He has directed much research in England including the project â€˜Effective Teachers of Numeracy in Primary Schoolsâ€™, and was deputy director of the five-year Leverhulme Numeracy Research Program, examining teaching, learning and progression from age 5 to age 11.