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The Bianchi Classification in the Schücking-Behr Approach[1]

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The historical development of the Bianchi classification of homogeneous cosmological models is described with special emphasis on the contributions by Schücking and Behr.

KEY WORDS: Bianchi models; Lie algebra; cosmology.3

1. INTRODUCTION

Today, the Bianchi classification of 3-dimensional Lie algebras is no longer presented by the original Bianchi method [4]. The now-common approach is usually credited to C. G. Behr, but one never sees any reference to a paper by him. Tracing the references back in time one is most often led either to the paper by Ellis and

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MacCallum [26] or to the paper by Estabrook, Wahlquist and Behr [25]. The latter, along with a brief summary of the (then-) new method, contains a promise that a separate paper by Behr should be published, where the approach would be presented in more detail. However, a thorough search through the author indexes in *Physics Abstracts* for 1968 and the following years has revealed no entries by Behr.

At this point, this Editor (A. K.) launched an investigation into the mystery. It turned out that Christoph Behr left research in physics soon after 1968 and has never published the planned paper. Today, he is retired and lives in Stuttgart, Germany, but has not preserved any notes and was not able to reconstruct the original account. George Ellis recalled that he first learned the approach from notes taken by Wolfgang Kundt at the relativity seminar in Hamburg. The actual inventor of the approach turned out to be Engelbert Schücking.

Several e-mails followed. W. Kundt has found his old notes and has kindly agreed to translate them to English and write them up for publication. This text is published below for the first time (and, as every newly published text, was approved for publication by two referees). This is the closest approximation to the original source that was possible to reconstruct today. Here follow short personal recollections on this subject of the main players in this story: C. Behr, E. Schücking, F. B. Estabrook with H. D. Wahlquist, G. Ellis and W. Kundt, and editorial comments written by Robert Jantzen. Their titles were invented by H.-J. Schimidt.

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by Andrzej Krasiński

2. DEGENERATIONS OF LIE ALGEBRAS

The approach to the Bianchi classification in question goes back to 1966 and 1967, when I was working as Postdoctoral Resident Research Associate at the Physics Section of the Jet Propulsion Laboratory at Pasadena, California. I belonged to a small group of relativists headed by Frank B. Estabrook. During that time I carried out research in relativistic cosmology. Emphasis was put on spatially homogeneous world models (i.e. on space-times admitting of groups of motions that are transitive on space-like 3-dimensional orbits). In addition to this I considered solutions of Einstein's field equations that allow for groups of motions which are transitive on the whole space-time manifold. As suggested by and frequently discussed with Professor Engelbert L. Schücking, then at the University of Texas at Austin, I investigated transitions between cosmologies with non-isometric groups of motions by means of singular transformations. This research was motivated by the wish to get insight into the physical and geometrical properties of, and relations between, known solutions of the field equations. Further on I hoped that new solutions could be constructed by this method.

The "Bianchi-Behr classification" was the first result of this research. The then-new approach is presented to some extent in the paper entitled "Dyadic analysis of spatially homogeneous world models," published by Frank B. Estabrook, Hugo D. Wahlquist and myself [25] in 1968.

About 1967 I had in mind to publish a paper containing the then-new approach to Bianchi classification in detail, together with some results in cosmology I hoped to obtain. Later on I gave up this plan after having encountered considerable difficulties, which prevented me from coming to the desired cosmological results.

In December 1966 I gave a seminar on "Degenerations of Lie Algebras" at the University of Texas at Austin. Further on, I gave a talk entitled "Contractions of Lie groups and general relativity" at the 1967 Winter Meeting of the American Physical Society in Pasadena, California. In 1968 I left the Jet Propulsion Laboratory—and relativistic cosmology.

Since about 1972 I have been no longer engaged in relativity.

At Austin, Texas, in the summer of 1966, Engelbert Schücking informed me about his original idea for the classification of 3-dimensional Lie algebras. If my memory serves me right, Engelbert Schücking did this in the way as it is stated by Wolfgang Kundt in the present document (*Schücking's Program, step 2*). Then I left Austin for Pasadena. I certainly agree that my approach to the Bianchi classification [25] should not be called "Bianchi - Behr classification"—but "Bianchi - Schücking - Behr classification", as suggested by Robert Jantzen in his editorial comments (section 6 of the present text).

by Christoph G. Behr

3. CLASSIFICATION OF LIE ALGEBRAS

In 1955 I was hired by Otto Heckmann, director of the Hamburg Observatory, to work on a new edition of his book [13]. Heckmann had generalized the Newtonian cosmology of Edward Arthur Milne and William McCrea [10] by considering also linear anisotropic flows of the substratum with velocity v_j, Euclidean coordinates x_k and time t given by

$$v_j = A_{jk} x_k, \qquad A_{jk} = A_{kj}(t); \qquad j, k = 1, 2, 3.$$

I knew about Kurt Gödel's model of a rotating universe in Einstein's theory of gravitation [14] and showed him the exact Newtonian analog of Gödel's model. This made it clear that a global Newtonian flow could also show rotation and that the restriction to symmetric A_{jk} was not warranted. The existence of a Newtonian analog to Gödel's relativistic model suggested the possible existence of relativistic analogs to Heckmann's anisotropic Newtonian models and he allowed me to work on their construction. This led at once to the integration of the Bianchi type I models with Euclidean space sections. I did not know of Edward Kasner's [7] solution and Abraham Taub's [15] model for the vacuum case. But it was clear that there could be other models with homogeneous 3-spaces which were not flat.

In Howard Robertson's article on cosmology [9] I found a reference to Luigi Bianchi and got hold of Bianchi's text [6] which discusses also homogeneous 3-dimensional Riemannian manifolds. The original paper of 1898 I saw only many years later when Remo Ruffini gave me a copy of it. I knew, possibly from the book by Sophus Lie and Georg Scheffers [3], a classification of complex 3-dimensional Lie algebras, but that was not much help for the real case.

The derivation of Bianchi's classification was a necessary step in attempting the construction of the anisotropic dust models with homogeneous space sections. Only a few results of these calculations were published [18], [19]. The basic technique was using the automorphism group of each Lie algebra, i.e. the homogeneous linear transformations which leave the third rank tensor of the structure constants fixed, in a similar way as one employs the Galois group for the solution of algebraic equations.

Gödel showed me in 1961 that he had done all models for each Bianchi type separately. When I asked him when he was going to publish his results, he said: "not in the next ten years." He never did and this full drawer in one of his filing cabinets still waits for an editor. I expect that one will find there the most elegant version of the Bianchi classification. He was one of the most ingenious mathematicians.

Wolfgang Kundt's paper certainly gives the gist of my seminar talk in 1957 but emphasis and telegraphic style of proofs are certainly his. It is possible that my own notes are still extant, but they would be in boxes that have not been looked into for forty years and retrieval might take some time. Anyhow, I am grateful to Wolfgang for his reconstruction.

I had helped Christoph Behr with his "Diplomarbeit" (Thesis, Hamburg 1960). In 1965 and 66 Christoph spent some time at the Relativity Center in Austin, Texas. He was looking for a problem and I suggested we derive the homogeneous models from a simplified variational problem. This had been done by Hermann Weyl for the Schwarzschild-Droste solution in his book [17][10]. The idea is to integrate over a right- or left-invariant volume of the homogeneous submanifolds and vary the Lagrangean without disturbing the symmetry. One should then get a one-dimensional variational problem, exactly what Weyl got.

If I remember correctly, this worked only for vanishing vector of the structure constants (in Ellis and MacCallum's classification: class A). I was not able to

[10] Editor's note: Droste's original paper [5] will be reprinted in this series. It will be seen that Droste himself used the variational approach.

understand why not in the other cases and, therefore, did not publish the result.[11] I suppose it would have been at that occasion that I familiarized Christoph Behr with the classification of the three-dimensional structure constants of real Lie algebras. He then went to Caltech to work with Frank Estabrook and Hugo Wahlquist.

By the way, the problem of simplified variational problems is not without interest, especially for theories with internal symmetry spaces. Alex Harvey once showed me a paper by the Brandeis mathematician Richard Palais who called it the "principle of minimal criticality." His theorem says that it sometimes works and sometimes not, but I was unable to follow his higher mathematics.

The appendix A of MacCallum's magisterial paper on the classification of 4-dimensional Lie algebras [32] should be reprinted here. It is, among other things, a remarkable piece of historical detective work.

by Engelbert Schücking

4. MIXMASTER MODELS AND OTHERS

Directly motivated by the successes of the Newman-Penrose tetrad formalism in problems where there are preferred null congruences, Estabrook and Wahlquist in 1963 developed a complete 3+1 formalism for orthonormal tetrad fields aligned with a preferred timelike congruence. We adopted symbols for vectors and tensors in the local orthogonal 3-space using the terminology customary in fluid dynamics, and called the result "dyadic analysis," a la Gibbs. We were able to discuss this approach with participants at the First (1963) and Second (1964) Texas Symposiums, among whom were some of the contributors to this Golden Oldie. Our paper on dyadics appeared in print in J. Math. Phys. in 1964 [20].

As a result of those interactions, we exchanged letters with and sent a preprint to G. F. R. Ellis, who shortly afterwards sent us a copy of his Cambridge thesis, in which he noted some of our results. It was immediately obvious that we had been treading parallel paths, while using very different formalisms. In the last section of his thesis, Ellis applied his formulation to homogeneous dust cosmologies, not, however, employing the Bianchi-Behr classification scheme of which he was unaware at the time. As Ellis points out in his comment here, it is quite surprising that Engelbert Schücking, who apparently was in possession of this superior classification in 1957, never published it, nor utilized it in his subsequent contributions to cosmology.

Schücking himself proposed that Christoph Behr join us at JPL as a NAS/NRC postdoc, and when he arrived in September, 1966, Christoph showed us the group theoretic approach to homogeneous cosmologies he was pursuing. Since a preferred timelike congruence is available, and the three spacelike orthonormal triad vectors can be invariant under the isometry group action (so their Ricci

[11] Editor's note: The reason is nicely explained by M. MacCallum in Ref. [30].

rotation coefficients are only time dependent), it was for us a natural application of the $3 + 1$ dyadic formalism. We were able to treat these cosmologies using 1st order equations for the dyadic variables, rather than 2nd order equations for the metric coefficients, as had been the customary approach. Doing so accomplished a great simplification of the evolution equations, and we also discovered that the Jacobi identities appeared as the simple eigenvector equation, $\mathbf{n}.F = 0$, where the symmetric dyadic F and the 3-vector **n** comprise the 9 "structure constants" of the algebra (but here treated as time-dependent Ricci rotation coefficients, or affinity fields).

Orienting the spacelike orthonormal triads to diagonalize F, one is immediately led to a classification of sub-cases somewhat different than Bianchi's, simpler and more natural. We grouped them into two classes, vanishing **n** and non-vanishing **n**, later designated types A and B, respectively, by Ellis and MacCallum [26]. We introduced the subscript h notation into the continuous types VI and VII; Bianchi type III became type VI with $h = -1$. Christoph was already familiar with this purely algebraic structure, which appeared quite naturally as a constraint in the dyadic differential equations, so we agreed to call the resulting classification of types Bianchi-Behr types. The most significant consequence of combining this (to us new) algebraic classification with the 1st order dyadic equations was the quick proof of the general conservation law for Bianchi-Behr types, including h. It was published in the joint paper with Behr [25].

We also worked out the consequences for a few of the simplest cases. Type IX had been carefully treated by Behr [22] using the 2nd order metric equations, but in the diagonalized orthonormal frame the symmetric sub-type studied by Gödel [16] was particularly elegant. For vacuum, or with cosmological constant, a quartic Hamiltonian emerged which anticipated Misner's mixmaster models.

Preprints of our cosmology paper were circulated at the Third Texas Symposium (in New York, January 1967). We remember a long interaction there with Wolfgang Kundt, who was most interested in our local proof that, regardless of the matter content and its behavior (excluding singularities), Bianchi-Behr type is conserved. After his return to Hamburg he wrote that, working with Bernd Schmidt, he now understood this group-theoretically.

At the London GRG meeting in the summer of 1968 after our paper appeared, we learned that, again, closely parallel work had been underway at Cambridge by Ellis and MacCallum [26], and that they had found the same type classification to be of great value. As is frequently the case in relativity, new developments may have several independent sources. In any event, to Chris Behr belongs the credit for initiating and motivating our contribution with him, where the Bianchi-Behr classification was first published in a refereed journal article and used to derive significant new results.

by F. B. Estabrook and H. D. Wahlquist

5. TETRAD METHODS

The present streamlined understanding of the relationship between the Lie algebra structure constants and the Bianchi group types, needed to study systematically the dynamics of spatially homogeneous universe models, was developed by Engelbert Schücking in Hamburg in the years 1956 to 1957, and became general knowledge through the papers of Estabrook, Wahlquist, and Behr (1968) [25] on the one hand and of Ellis and MacCallum (1969) [26] on the other. Both sets of authors derived their understanding of this topic from Schücking. The implicit or explicit use of Lie groups to characterise spatially homogeneous geometries invariant under a G3 of isometries, from now on referred to as Bianchi geometries, actually dates from much earlier, but the method of classification did not arrive at its present concise form until the work of Schücking.

Already in 1921 E. Kasner [7] looked at the Bianchi I (abelian) spatially homogeneous anisotropic models in the vacuum case and in 1933 G. Lemaitre [8] examined the matter case, but neither did so from a group theory viewpoint - these are obvious anisotropic generalisations of the Robertson-Walker models that did not require group theory in their derivation. Study of their dynamics was later picked up by Schücking, B. B. Robinson, A. K. Raychaudhuri, K. S. Thorne, and others, but usually without specific group theory characterisation.

In 1935-1936, H. P. Robertson [11] and A. G. Walker [12] in discussing the Robertson-Walker models gave generic Lie group prescriptions for obtaining these solutions, in effect treating them as Bianchi geometries, but apparently never used specific Lie algebra types in their calculations. In particular they did not identify the group types that applied to these models, nor explain how the simply transitive and multiply transitive subgroups were related to each other. Their actual derivation of the metrics rather relied directly on their spatial homogeneity and isotropy, cf. [9].

The first systematic use of Lie group theory to study Bianchi geometries was by Kurt Gödel, who developed the study of the geometry and dynamics of Bianchi IX cosmological models in 1952 [16] as a follow up to his stationary space-time homogeneous model (the Gödel universe of 1949 [14]). Thus he was the first to introduce these anisotropic models explicitly characterised by their group symmetries, but did so in an elusive and enigmatic way, to a considerable degree relying on the special properties of Bianchi IX symmetries i.e. the SO(3) symmetry group. However according to Schücking (who visited him in Princeton) he carried out many calculations for more general group types, but never published them (details are probably still extant in his notes)[12]. Then Taub (1951) [15] derived the dynamic equations for the generic vacuum Bianchi geometries, explaining the techniques needed to develop these equations in a non-holonomic and

[12] For more on Gödel's contribution to relativistic cosmology, see G. F. R. Ellis, the editorial note preceding Refs. [14] and [16].

non-orthogonal basis, thus giving the first easily available systematic exposition of methods usable for all the spatially homogeneous Bianchi models. Somewhat later Schücking developed a similar version suitable for cosmology (he derived the equations with a fluid source term) and published a brief note on it in the 1958 Solvay conference proceedings [18], and a detailed account in the Appendix to the Heckmann and Schücking article in the Witten volume (1962) [19]. He emphasized the role of the automorphism group in conjunction with use of a non-orthogonal basis in simplifying the equations, without giving details. This was taken up much later by Jantzen and others. The Heckmann and Schücking article however did not give the specific structure constant decomposition that is used today. It should be mentioned that A. Z. Petrov in Russia also carried out an independent systematic examination of space-times invariant under Lie groups, resulting in a major book [21], but he did not focus specifically on cosmological models.

When working on my PhD thesis in Cambridge from 1961 to 1964, on advice from Dennis Sciama I investigated first homogeneous (steady state) anisotropic space-times and then Bianchi models. In doing so I developed a generic $1 + 3$ tetrad approach, working from geometric ideas in Helgason's text on Differential Geometry and Symmetric Spaces, supplemented by knowledge of Lie algebras from Cohn's book on the topic and the detailed study of Continuous Groups of Transformations in Eisenhart's book of that name. I was also given helpful advice by Schücking when he visited Hermann Bondi and Felix Pirani at King's College, London, in about 1961; this gave me access to a typewritten version of the crucial 1962 article in the Witten book. The orthonormal tetrad formalism I developed was utilised in my PhD thesis in 1964 and published in 1967 in a paper [23] that systematically examined all Locally Rotationally Symmetric ('LRS') dust spaces (invariant under multiply transitive groups of isometries). At about the same time C. Behr was working on Bianchi IX models in Hamburg and L. Shepley in Princeton was developing somewhat similar methods for all the Bianchi models, under advice from John Wheeler. The dynamics of these models were then studied by others, including P. T. Saunders, C. W. Misner (who introduced differential form methods), S. W. Hawking, and C. B. Collins. But none of these workers used the classification that is now in use. The 3-dimensional Lie algebras were rather classified in terms of the reduction to the canonical forms for their structure constants determined by Bianchi [4], made accessible by Taub's (1951) paper [15].

For me a key event was a visit that I and Graham Dixon made to the 1st Institute of Theoretical Physics in Hamburg in about 1968. We stayed with Wolfgang Kundt and his wife in Hoheneichen and took part in seminars with the Hamburg group, which did not then include Schücking, Ehlers, or Sachs (they were all in Texas by that time). One evening Wolfgang showed me his notes on the Schücking seminar that are reproduced here, and I spent a night poring over them. I believe I must have copied down a large part of them by hand, as was my custom in those days, which was before Xeroxes were available (and long before either word processors or Latex were on the horizon). The key element that I had not seen anywhere else was the representation of the structure constants given in the Lemma in Step 2 of Kundt's notes, its importance arising firstly from the fact that this immediately solved the Jacobi identities as shown in that Lemma, and secondly that it consequently gave such a simple classification of the Lie Algebras on using the obvious diagonal bases (in contrast to the numerous pages of projective geometry used by Lie and Bianchi). Engelbert now comments that the representation used was obvious as it was just a reduction to irreducible parts, but this step is only 'obvious' once one has raised the lower two indices on the structure constants by use of the antisymmetric 3-symbol to give a quantity with two upstairs indices.

On my return to Cambridge I immediately adapted my $1 + 3$ tetrad methods to this formalism and worked out the dynamical consequences for Bianchi models with the help of Malcolm MacCallum, who became my PhD student. Thus in effect Ellis and MacCallum (1969) [26] was a development of Ellis (1967) [23] in the light of Schücking's seminar paper, as reported by Kundt. This work was developed independently of the Estabrook-Wahlquist-Behr 1968 paper [25], which appeared shortly before ours did and contained much overlapping material in a completely different formalism. Our paper included some other aspects however, giving the non-diagonal form that is useful in some cases, giving a complete classification of the relation of the Bianchi models to the multiply transitive spatially homogeneous models and in particular showing that $k = -1$ RW models corresponded to Bianchi VII (and not Bianchi VIII as many had believed up to that time),[13] and showing the exceptional status of the Bianchi VI_h models with $h = -1/9$. We also introduced the Class A/Class B notation that is now in common use. This orthonormal formalism for Bianchi models later became much more powerful through introduction of an expansion-normalised version of the variables by John Wainright, providing the basis for systematic use of dynamical-systems methods to illuminate the dynamics of these models (see [31] for an exposition).

The puzzle is that the Schücking seminar recorded in the note following, containing the crucial representation and resulting Lemma together with the classification of the 3-d Lie algebras and details of how this classification corresponds to that given by Bianchi, took place in January 1957. But Schücking did not include this material in the 1962 Witten article, nor mention it to me when I visited him in London at about that time. It seems strange that such an important feature was not included in that article, and so was not for example mentioned in either my own PhD thesis in 1964 nor in Shepley's thesis in 1965. We have here an enigmatic aspect of Engelbert - he had the key to the best representation of the Bianchi structure constants already in 1957, but chose not to publish it even when writing in detail about these spaces.

[13] This had been independently discussed by Grishchuk [24], in a paper unknown to us at the time.

He has explained to me now that he was planning to present these results in a book which never got written for two reasons: first he did not attain the deeper insight into the mechanism of Einstein's gravitation he had hoped for by using these methods, and second he believed that what he had done was already contained in unpublished work of Gödel's. Personally I doubt that – Engelbert's work was developed in the conducive environment of the Hamburg groups consisting of P. Jordan, O. Heckmann, E. Schücking, J. Ehlers, W. Kundt, and latterly R. Sachs and M. Trümper, where elegant covariant methods were communally developed and treasured. The development of Schücking's classification method was perhaps natural in that context, but it was not obvious to other workers in the field at the time, and I am not aware of any evidence that Gödel developed this particular insight. In any case I was deeply indebted to Wolfgang Kundt for showing me the seminar notes on Engelbert's work, for they made my paper with Malcolm MacCallum possible, which then became a useful source for many other workers.

by George F. R. Ellis

6. DIAGONAL FORM

To give the document some context as briefly summarized in [25] and its reference [19], Gödel had first used Bianchi type VIII and IX dust models in cosmology in 1949 and 1952 [14], [16], while Taub had methodically investigated all the vacuum Bianchi models in 1951 [15], giving the form of the spacetime metric and the Killing vector fields for all Bianchi types explicitly. This was followed up by work on dust models by Heckmann and Schücking [19], reported here as Schücking's program then in progress in 1957. In this present document, the diagonal form of the structure constants introduced by Ellis and MacCallum [26] (in which the symmetric part of the structure constant matrix is diagonal and the vector is aligned with one of the basis vectors) is suggested but not implemented. It is this more convenient choice of representatives which was described in the Behr article and later universally used in Bianchi cosmology which has come to be known as the Bianchi-Behr classification, but should clearly now be called the Bianchi-Schücking-Behr classification.

The one remaining detail not discussed in the literature is the actual relation between Bianchi's canonical choice of structure constants appearing in the article below and the logical diagonal form choices in the Ellis-MacCallum scheme. In their notation

$$C^{\alpha}{}_{\beta\gamma} = \epsilon_{\beta\gamma\delta} n^{\beta\delta} + a_{\delta}\delta^{\delta\alpha}_{\beta\gamma}, \quad [e_{\beta}, e_{\gamma}] = C^{\alpha}{}_{\beta\gamma} e_{\alpha},$$
$$C^{\alpha\delta} = C^{\alpha}{}_{\beta\gamma}\epsilon^{\delta\beta\gamma} = C^{(\alpha\delta)} + C^{[\alpha\delta]} = n^{\alpha\delta} + \epsilon^{\alpha\delta\gamma} a_{\gamma},$$

where

$$n^{\alpha\delta} = C^{(\alpha}{}_{\beta\gamma}\epsilon^{\delta)\beta\gamma},\ a_\gamma = \frac{1}{2}C^\alpha{}_{\gamma\alpha},\ n^{\alpha\delta}a_\delta = 0,\ a_\alpha a_\beta = \frac{1}{2}h\epsilon_{\alpha\gamma\mu}\epsilon_{\beta\delta\nu}n^{(\gamma\delta)}n^{(\mu\nu)},$$

the last relation first given by Collins and Hawking [28], apparently due to an observation of Hawking. The invariant constant h which appears in it was first introduced by Behr and is related in the following way to the constant h which appears below in the Bianchi classification scheme for Bianchi types VI and VII, call it h_B for Bianchi's parameter: $h = -(h_B+1)^2/(h_B-1)^2$ for type VI ($h \le 0$, $-1 \le h_B < 1$, $h_B \ne 0$) and $h = h_B^2/(4-h_B^2)$ for type VII ($h \ge 0$, $0 \le h_B^2 < 2$). For type VI the parameter values $h_B = 0$ ($h = -1$) for Bianchi type III and $h_B = 1$ ($h \to -\infty$) for Bianchi type IV are excluded. The class A type VI corresponds to $h_B = -1$ ($h = 0$), while the class A type VII corresponds to $h_B = 0$ ($h = 0$). The modern designation for these types is in fact VI_h ($h \le 0$) and VII_h ($h \ge 0$) with III = VI_{-1}, and VI_0 and VII_0 denoting the class A subcases.

The diagonal form closest to Bianchi's scheme is $(n^{\alpha\beta}) = \mathrm{diag}(n^{(1)}, n^{(2)}, n^{(3)})$, $(a_b) = (0, 0, a)$, $an^{(3)} = 0$, so that $a^2 = hn^{(1)}n^{(2)}$ (Behr's definition of h) and

$$(C^{\alpha\beta}) = \begin{pmatrix} n^{(1)} & -a & 0 \\ a & n^{(2)} & 0 \\ 0 & 0 & n^{(3)} \end{pmatrix} \leftrightarrow \begin{array}{l} [e_2, e_3] = n^{(1)}e_1 - ae_2 \\ [e_3, e_1] = ae_1 + n^{(2)}e_2 \,. \\ [e_1, e_2] = n^{(3)}e_3 \end{array}$$

The ordered diagonal values $n^{(\alpha)}$ can be taken as $1, \ldots, -1, \ldots, 0, \ldots$ as suggested by Schücking, while a is either zero (class A or unimodular) or not zero (class B or nonunimodular), in which case it can be made 1 except for Bianchi types VI and VII. The Bianchi canonical choices for VI and VII instead make the first 2×2 subblock of the matrix $(C^{\alpha\beta})$ respectively offdiagonal and upper-triangular. The transformation between the choices is mediated by

$$X_\alpha = e_\beta A^{-1\beta}{}_\alpha,\ \bar{n}^{\alpha\beta} = \det(A^{-1})A^\alpha{}_\gamma n^{\gamma\delta}A^\beta{}_\delta,\ \bar{a}_\alpha = a_\beta A^{-1\beta}{}_\alpha,$$

where the bar indicates a Bianchi basis X_α and its structure constants $\bar{C}^\alpha{}_{\beta\gamma}$.

For type VI with $q = n^{(1)} = -n^{(2)} = 1$, Ellis and MacCallum [26] introduced a second canonical choice for which A is a rotation by angle $\pi/4$ about the third direction leading to the nonzero 2×2 block of $[C]$ of the matrix $(C^{\alpha\beta})$ taking the form

$$[\bar{C}] = \begin{pmatrix} 0 & q-a \\ q+a & 0 \end{pmatrix} = \begin{pmatrix} 0 & h_B \\ -1 & 0 \end{pmatrix}, \qquad [A] = 2^{-1/2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

leading to the above relationship between the parameters. The same transformation about the first basis vector can be used in the type VIII case with $q = n^{(1)} = n^{(2)} = -n^{(3)} = 1$ to get a Bianchi basis apart from a factor of 2 and a permutation. (These lead to null bases on the dual space for the indefinite quadratic form associated with $n^{\alpha\beta}$ in this 2×2 block.)

For Bianchi type VII with $q = n^{(1)} = n^{(2)} = 1$, the transformation is instead a scaled Lorentz boost in the 2×2 block

$$[\bar{C}] = \begin{pmatrix} -q & qh_B \\ 0 & -q \end{pmatrix}, \qquad A = \begin{pmatrix} -1 & \tanh 2\theta & 0 \\ \tanh 2\theta & -1 & 0 \\ 0 & 0 & -\cosh 2\theta \end{pmatrix},$$

where $\sinh 2\theta = h^{1/2} = q/a$ and $\tanh 2\theta = \sqrt{h/(1+h)} = h_B/2$.

Apart from permutations and possible scale factors and signs, this explains the relationship between the canonical choices made by the modern classification and those of Bianchi, which were really those of Lie who used the first seven Roman numerals in reverse order compared to Bianchi to describe the equivalence classes of 3-dimensional complex Lie algebras [1, 2]. Bianchi gives his own derivation of Lie's classification with his refinement to the real case in his own book [6], using the dimension of the derived Lie algebra (equivalent to the rank of $(C^{\alpha\beta})$) and its structure to arrive at canonical bases for each type (see [33] for an English translation and commentary). The same paper of Bianchi Remo Ruffini had given Schücking we translated together in 1973 when I was a Princeton undergraduate, just after MacCallum had been a postdoc with Taub explaining in [27] the variational problems referred to above by Schücking, and I then had the pleasure of being Taub's final graduate student before his retirement in 1978. A single parametrized formula giving a coordinate representation for all the diagonal form Lie algebra vector field bases as a function of the four parameters $(n^{(1)}, n^{(2)}, n^{(3)}, a)$ came out of that experience [29].

by Robert Jantzen

7. HOMOGENEOUS COSMOLOGICAL MODELS

In 1956/7, Engelbert Schücking was a young collaborator of Otto Heckmann at Hamburg's Observatory in Bergedorf, and at the same time a member of Pascual Jordan's Seminar on General Theory of Relativity at Hamburg's I. Institute for Theoretical Physics, together with Jürgen Ehlers and myself as the two junior members. I had finished my diploma in October 1955 and still tried to get a better understanding of Einstein's Theory of Gravitation, of Quantization, and of various branches of Mathematics. Engelbert occasionally confronted Jürgen and me with his most recent studies—from all disciplines of physics—in *spontaneous seminar* presentations. One of those is the subsequent approach to a determination, in more or less closed form, of all spatially homogeneous cosmological models, of which I took notes which I edited in polished form at home, in January 1957, mostly in (German) handwriting because of the many special symbols and equations whose typing would have had to be very incomplete anyway. It was my tacit understanding that Engelbert, or one of his younger associates—like his later student Christoph Behr—would take the matter into print, i.e. that my writeup had no other purpose than to help me remember this interesting work. Apparently, no such documentation can be easily accessed today.

Schücking's idea was that the class of known exact solutions of Einstein's field equations, applicable to Cosmology, could be vastly extended (over Friedmann and Lemaître's isotropic class) under the assumption of *spatial homogeneity* because the assumed translational symmetry of spacetime guarantees the existence of 3-d subspaces which are homeomorphic to the respective Lie group of isometric mappings. Spatial homogeneity is, of course, not an exact property of our world, but should be a rather realistic approximation to the Universe on length scales above those of the largest clusters of galaxies, some 10^2Mpc and beyond. A large candidate class of analytic models for our Universe would help understand its large-scale properties and expected evolution in time. How to achieve this goal?

By Wolfgang Kundt

8. ENGELBERT SCHÜCKING – BRIEF AUTOBIOGRAPHY[14]

When teaching history of astronomy (and astrology) on television I have been advertized as a "Westphalian Gemini" since I was born in that part of the European Union under this sign on May 23, 1926. By age 15, I had contributed 368 days of sunspot observations[15] to the Zürich Sunspot-statistics and realized that there are more intelligent ways to do science. In fact, when I was 12 and decided I needed to learn calculus to determine the orbit of comets, Fräulein Killing, daughter of the Münster mathematician, loaned me a book. The work of her father who first classified semi-simple Lie algebras and discussed symmetries of Riemannian manifolds has dominated my life in science.

I studied at the universities in Münster, Göttingen and Hamburg and got a Dr. of Science degree in mathematics from Hamburg University (a physics degree required a lab in analytic chemistry which I arrogantly refused to take). In Hamburg I worked for Pascual Jordan who had just become interested in relativity and taught me that people with irresponsible ideas in politics can be decent personally. I had taught myself relativity from books by Arthur Eddington and Hermann Weyl, helped Jordan develop his variation of Theodor Kaluza's theory, and investigated with Otto Heckmann the structure and solutions of Newtonian cosmology. I then started the investigation of anisotropic, spatially homogeneous dust models in relativistic cosmology. That led, in collaboration with István Ozsváth, to

[14]Editor's note: A more extended autobiography of E. Schücking may be found in *On Einstein's Path*, Ed. A. Harvey, Springer 1999, pp. 1 – 14. Further historical remarks are in the biographical notes by A. Harvey (pp. vii-viii) and the article by I. Ozsvath (pp. 339-352) in the same volume.

[15]Astronomische Mitteilungen, ed. by W. Brunner, 140, 141, 142, Zürich, Schulthess and Co. 1941, 1942, 1943.

counter-examples of some versions of Ernst Mach's principle in Einstein's theory of gravitation.

I spent five years at the University of Texas at Austin and started in 1963 with Alfred Schild and Ivor Robinson the Texas Symposia on Relativistic Astrophysics which continue every other year, the next one will be in Florence, Italy in December 2002.

In 1967 I moved to New York University and supervised some twenty theses in relativity (mostly cosmology). More recent work is a book on homogeneous gravitational fields with Eugene Surowitz, the study of the Cabibbo-Kobayashi-Maskawa manifold with István Ozsváth and Jerome Epstein, and an attempt to geometrize the electro-weak interaction, called the "sub-standard theory." I am also interested in the history of science, and I help in translating Einstein's papers into English.

by Engelbert Schücking

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This textbook is the first of its kind in Kazakhstan to be devoted to the theory and practice of foreign language education. It has been written primarily for future teachers of foreign languages and in a wider sense for all those who are interested in the question of the study and use of foreign languages. This book outlines an integrated theory of modern foreign language learning (FLL) which has been drawn up and approved under the auspices of the school of science and methodology of Kazakhstanâ€™s Ablai Khan University of International Relations and World Languages.