ADDENDUM TO THE PAPER "CENTRAL QUADRICS IN THE EUCLIDEAN n-SPACE"

ÁKOS G.HORVÁTH

Abstract. In this paper we give some proofs on the topic of the paper "Central quadrics in the Euclidean n-space". For example we give a more detailed description of the construction of Chasles and the wire (thread) model of Staude, respectively.

This paper is not self-contained. I suggest to look at the paper [3] for the necessary definitions and theorems.

1. Apollonius

Theorem 1 (Theorem of Apollonius). *An ellipsoid in the Euclidean n-space has n associated scalars invariants with respect to any affinity of the space. It can be defined by the formulas:*

$$v_k := \sum_{i_1<i_2<\cdots<i_k} \det G[x^{i_1},\cdots,x^{i_k}], \tag{1}$$

where $\{x^1,\dots,x^n\}$ is a complete conjugate system of semi-diameters and $G[x^{i_1},\cdots,x^{i_k}]$ is the Gram matrix of the vector system $x^{i_1},\cdots,x^{i_k}$.

Proof. Consider the matrix X^TX+tI where t is a real number and I is the identity matrix. Then we have

$$\det\left[(X^TX+tI)\right]=\det[A^2+tI]=\sum_{k=0}^{n} t^{n-k}\prod_{i_1<\ldots<i_k} a_{i_1}^2\cdots a_{i_k}^2.$$

On the other hand, if $X^TX=:G=[g_{i,j}]$ is the Gram matrix of X, and $G_{i_1,\ldots,i_k}$ denotes its k-minor defined by the elements in the crossing of those rows and columns which correspond to the multi-index $\{i_1,\ldots,i_k\}$, than using an inductive argument and the rule on the addition of determinants we can prove that

$$\det\left[(G+tI)\right]=\sum_{k=0}^{n} t^{n-k}\sum_{i_i<\ldots<i_k}\det[G_{i_1,\ldots,i_k}].$$

These two formulas imply that for all sets of complete conjugate systems hold the equality

$$v_k=\prod_{i_1<\ldots<i_k} a_{i_1}^2\cdots a_{i_k}^2$$

which gives our statement. □

Statement 1 (The Apollonian pedal curve). *Consider the set of homothetic ellipsoids $E(\lambda)=\{x\in\mathbb{R}^n : \sum_{i=1}^n (x_i/\lambda a_i)^2=1\}$ $\lambda\in\mathbb{R}^+$ and a point $P=(u_1,\ldots u_n)^T$ of the space. The locus of those points of the ellipsoids which are nearest to P is a curve of order n which we call* Apollonian curve *of the system of the homothetic system.*

Remark 1. If $n=2$ the above Apollonian curve is the so-called Apollonian equilateral hyperbola. As we know from the book [1] it is also the locus of the centers of the elements of that pencil of conics which determined by the given ellipse E and a fixed circle which center is P and intersects E in four points. An immediate proof of this fact is the following. If a point Q of the plane is a center of a conic of the pencil, then its polar lines with respect to the given ellipse and circle are parallel to each other. In fact, for any point Q there is a point Q' in the projective plane that the polar of Q with respect to the element of the pencil are go through Q'. Since the polar of Q is the ideal line if it is a center of a conic from the pencil, hence the point Q' is also an ideal one. Since the polar of a point with respect to a circle is perpendicular

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to the segment connecting the point with the center of the circle this common direction of the two polar lines is known. Hence the locus does not depends on the radius of the circle, it only depends on the ellipse and the given point. Consider now such a center Q which is on the ellipse E. The polar q of Q with respect to E is the tangent of E at Q hence it is perpendicular to QP showing that Q is on the Apollonian hyperbola $\mathcal{A}_2(\tau)$. Clearly, $\mathcal{A}_2(\tau)$ contains also the center of E and the point P. We prove that the locus is a conic implying that it is agree the Apollonian hyperbola as we stated. In fact, if Q' and R' are the ideal points corresponding to the above connection to the centers Q and R, respectively, the polars of Q' and R' with respect to the conics in the pencil go through Q and R, respectively. If S' any further (ideal) point of the line $Q'R'$ then its polar s'_Q with respect to the conic with center Q and its polar s'_R with respect to the conic with center R intersect in a point S which is conjugate to S'. Of course S is a center of a conic of the pencil. In this way, we got two pencils of lines with support Q and R are in a projective (but not perspective) connection and the corresponding elements intersect each other in the points S of the searched locus, hence the locus by the Steiner's definition of conics is a conic. Observe that those ideal points which are giving by the directions of the axes of the ellipse are points of the locus, too since their polar lines with respect to the ellipse and the circle are parallel to each other. Hence the above conic is such a hyperbola which has (at least) five common points with the Apollonian hyperbola implying that the two conics are agree.

2. Famous applications in dimension three

The formulas of the previous sections gives a good frame to prove some known results in the three-space. In this section we examine three such problems.

2.1. Right cones and focal conics. The following theorem can be used in mechanics, descriptive geometry and pure geometry of the three-dimensional space, respectively.

Theorem 2 (See e.g. [6]). *Let $n = 3$. The locus of the apices of right cones through a real focal quadric is the other real focal quadric.*

Proof. To find the locus of the vertices of right cones which can envelope a focal quadric we consider the canonical equation of the cone through to C_k with apex x' ((27) in [3])

$$\sum_{i=1}^{n} \frac{x_i^2}{\left(s_k^i\right)^2} = 0.$$

It may represent a right cone if $n-1$ from the above coefficients are equals. Since for a fixed k the numbers $\left(s_k^i\right)^2$ are the distinct roots of the polynomial equation (10) in [3], two of them could be equal to each other if and only if $f((s_k^j)^2) = 0$ for an index $j = 2, \dots, n$ implying that it is the square of two consecutive semi-axes from $a_2, \dots, a_n$ which are equal to each other; hence either $(s_k^j)^2 = (a_j)^2 = (a_{j+1})^2 = (s_k^{j+1})^2$ or $(s_k^{j-1})^2 = (s_k^j)^2 = (a_{j-1})^2 = (a_j)^2$ hold. (In both of these cases the index k distinct from the other two ones.) This means that the vertex x' of the cone by definition is a real point of a focal quadric. In the first case, if $x'_j = 0$ then it is on C_j and if $x'_{j+1} = 0$ then it is on C_{j+1}. On the other hand if e.g. we assume that $\left(s_k^j\right)^2 = \left(s_k^{j+1}\right)^2$ $(j, j+1 \neq k)$ then the examined cone with vertex x' and through C_k has canonical form

$$C_k(x') : \quad \sum_{\substack{i=1 \\ i\neq j,j+1}}^{j-1} \frac{x_i^2}{\left(s_k^i\right)^2} + \frac{x_j^2}{\left(s_k^j\right)^2} + \frac{x_{j+1}^2}{\left(s_k^j\right)^2} = 0$$

showing that it has two semi-axes which are equal to each other.

In the three-dimensional case for $k = 3$ the index j may be 1 or 2. Since C_1 has no real point we know that C_2 contains the vertex of $C_3(x')$ and the latter is a right cone and vice versa. The equations of the two focal conics are

$$C_3 : \left\{ \frac{x_1^2}{a_1^2 - a_3^2} + \frac{x_2^2}{a_2^2 - a_3^2} = 1,\, x_3 = 0 \right\} \text{ and } C_2 : \left\{ \frac{x_1^2}{a_1^2 - a_2^2} + \frac{x_3^2}{a_3^2 - a_2^2} = 1,\, x_2 = 0 \right\}$$

and the canonical equation of the right cone through C_3 and with the vertex $x' \in C_2$ is

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_1^2} + \frac{x_3^2}{a_3^2} = 0.$$

□

2.2. **Chasles's construction on conjugate diameters.** The standard proof of Rytz construction based on the so-called "two-circle figure" (see in Fig. 1) in which we draw the incircle and the circumcircle of the ellipse. The same figure with a little modification enable to get another construction to solve this problem.

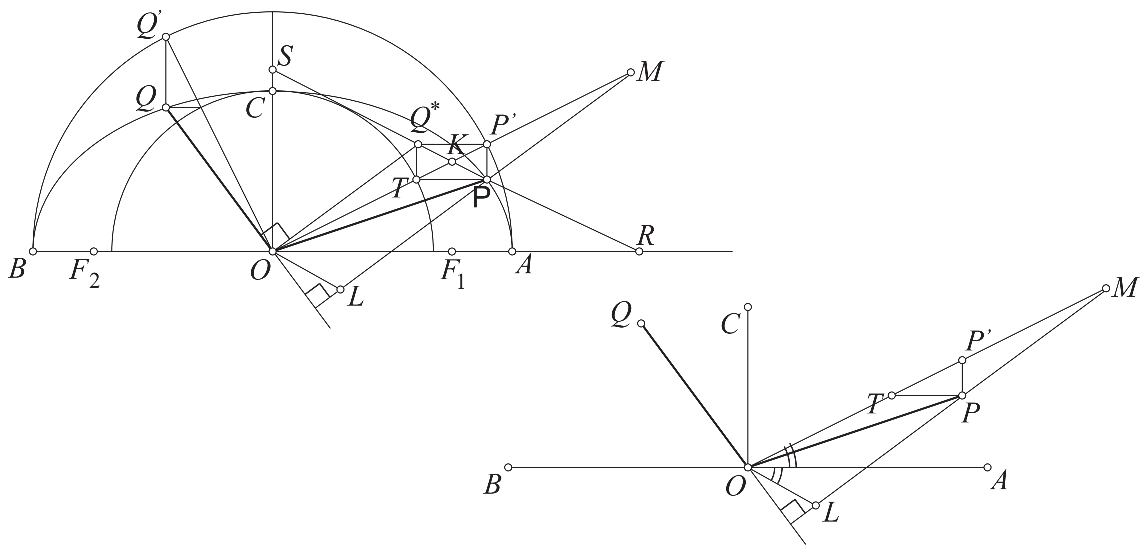

FIGURE 1. Conjugate diameters and axes

In fact, let the line PM be the normal of the ellipse at P. This is perpendicular to OQ hence it is parallel to $OQ^\star$. If M is the intersection of this normal with the line OK (as in the standard proof $Q^\star$ is the rotated copy of OQ by $\pi/2$ and K is the middle point of the segment $PQ^\star$), then the triangles $OTQ^\star$ and $MP'P$ are congruent to each other. From this we get that PM is congruent to $OQ^\star$. Let L be the reflected image of M at P. Since PL is parallel and equal to $OQ^\star$, OL is also parallel and congruent to $Q^\star P$. Hence the axis AB of the ellipse is the bisector of the angle $LOM\angle$. The other axis of the ellipse is perpendicular to AB at O, and if we draw parallels from P to these axes we get the points T and P' as the intersections of the line OM with these parallels, respectively. Clearly, the lengths of OT and OP' are equals to the lengths of the semi-axes, respectively. The construction has the following steps:

- Draw perpendicular to line OQ from P and determine the points M, L on this line by the property $|PM| = |PL| = |OQ|$.
- Draw the bisectors of the angle $LOM\angle$, these are the axes of the ellipse.
- Draw parallels to these bisectors from the point P and determine the intersections of these lines with the line OM (T, P'). The lengths of the semi-axes are the lengths of the segments OT, OP', respectively.

Observe that L and M determine a tangential pencil of confocal conics in the plane. From these pencils there are two conics an ellipse and a hyperbole through the point O. By the bisector property, the tangent lines of these conics at O are the axes of the searching ellipse. If we visualize the union of lines OM and OL as a cone with apex O and through the focal figure of the confocal system of conics determined by the points M and L then the axes of the ellipse are equals the axes of this cone. Hence this construction contains only such concepts whose analogous exist in higher dimensions, too.

Let us follow our investigations in the three-space. Our purpose to give a construction, which determine the axes of an ellipsoid from a complete system of conjugate diameters. To this consider the three pairwise conjugate semi-diameters OP,OQ and OR, respectively. The construction of Chasles (see in [2] or [7]) contains the following steps:

- Determine the three mutually perpendicular line through P, one of its the normal of the ellipsoid (perpendicular to the plane (OQR) at P) and the other two are parallel to the axes of the ellipse $\mathcal{E}$ which is the intersection of the plane (OQR) by the given ellipsoid. (By the above construction we can get these lines.)
- They contains the axes of a confocal system of quadrics dual to the confocal system of the given ellipsoid. By Statement 1 in [3] the axes of this systems can be given by the system $\{a_1^1, a_1^2, a_1^3\}$ as respective lengths of the semi-axes. The square of the semi-axes of the focal conics of this system

are $\{(a_1^1)^2-(a_1^3)^2,(a_1^2)^2-(a_1^3)^2\}$; $\{(a_1^1)^2-(a_1^2)^2,(a_1^3)^2-(a_1^2)^2\}$ giving an ellipse and a hyperbole, respectively. The square of the axes of $\mathcal{E}$ are $(\rho_2^1)^2=(a_1^1)^2-(a_1^2)^2$ and $(\rho_3^1)^2=(a_1^1)^2-(a_1^3)^2$. These values are known from the construction, hence we can give also the squares of the lengths of the semi-axes of the focal conics.

- The intersection of the two focal cones with apex O is the union of four common edges, the six planes determining by these edges intersect to each other in three mutual perpendicular lines which are the three axes of these two confocal cones. To prove this statement observe that the two focal cones are confocal quadrics and so they principal axes are common in their directions. The three principal axis intersects a plane of intersection in three points which are form an autopolar triangle of this plane with respect to the two intersection conics by the two focal cones. The four common points of the two conic of intersections determines that quadrangle which diagonal points form the only autopolar triangle with respect to both of the conics. By Statement 6 in [3] these are the searched axes of the given ellipsoid, while the planes through P and parallel to the principal planes of the ellipsoid cut off on these four lines parts equal in length to the semi-axes by Statement 7 in [3].

Observe that this theoretical construction cannot be constructed with ruler and compass in general, because the construction of the common lines of two cones of second order implies the construction of the common points of two conics, which is some cases unconstructible. More details on this problem can be found in [4].

2.3. **Staude's wire model in 3-space.** At the end of the eighteenth century O. Staude printed the book [8] with a lot of interesting results. H. D. Thompson wrote in his recension on the book the following: *Possibly the most interesting, although not the most general of the focal properties which Professor Staude deduces, are the following. (The "focal distance" from any point to the focus of the one of a pair of focal conics is defined as the shortest distance on the broken line to a point on that conic, and thence to the adjacent focus). In an ellipsoid, for any point on the surface, the sum of the focal distances to a focus of the focal ellipse and to the opposite focus of the focal hyperbola is a constant. And the normal to the surface at the point bisects the angle of the two lines. This theorem is the basis of the well-known wire model, No. 110 of the Brill collection, which represents the two focal curves of the system by wires rigidly connected, and which has a string fastened so as to represent the sum of the focal distances of the theorem.* This result can be cited without proof in the book of Hilbert and Cohn-Vossen [5], too.

Lemma 1. *Let $r(u_1,\dots,u_{n-1})=(x_1(u_1,\dots,u_{n-1}),\dots,x_n(u_1,\dots,u_{n-1}))$ be a hypersurface of the n-dimensional Euclidean space. F and P are two points of the space and Q is a point of the surface for which the length of the broken line FQP is minimal. Then the angles between the normal vector of the surface at its point Q and the respective segments FQ and QP are equal to each other.*

Proof. Denote the coordinates of F and P by $(f_1,\dots f_n)^T$ and $(p_1,\dots,p_n)^T$, respectively. We also denote by $f=\sum_{i=1}^n(x_i-f_i)^2$ and $g=\sum_{i=1}^n(-x_i+p_i)^2$ the functions $|FP|^2$ and $|PQ|^2$, respectively. Then the partial derivatives of the length function $\sqrt{f}+\sqrt{g}$ of variable $u_1,\dots,u_n$ have to vanish implying the equalities:

$$0=\frac{\frac{\partial f}{\partial u_i}}{2|FQ|}+\frac{\frac{\partial g}{\partial u_i}}{2|QP|}=\frac{\sum\limits_{j=1}^n(x_j-f_j)\frac{\partial x_j}{\partial u_i}}{|FQ|}-\frac{\sum\limits_{j=1}^n(x_j-p_j)\frac{\partial x_j}{\partial u_i}}{|PQ|}=\sum_{j=1}^n\left(\frac{(x_j-f_j)}{|FQ|}-\frac{(x_j-p_j)}{|PQ|}\right)\frac{\partial x_j}{\partial u_i}$$

for all $i=1,\dots(n-1)$. Hence $FQ/|FQ|-PQ/|PQ|$ orthogonal to $n-1$ independent vectors of the tangent hyperplane at the searched point Q meaning that it is parallel to the normal of the hypersurface at this point Q. Since the vectors $FQ/|FQ|$ and $PQ/|PQ|$ are unit vectors this proves the lemma. □

The following result can be found in [6], too.

Lemma 2. *Assume that H_1 and H_2 are two points on the same branch (on the different branches) of the focal hyperbola and E is any point of the focal ellipse. Then the difference $|H_1E|-|EH_2|$ (the sum $|H_1E|+|EH_2|$) of the lengths of the edges of the broken line H_1EH_2 is independent from the position of E. Similarly if E_1 and E_2 are two points on the focal ellipse then the difference $|E_1H|-|HE_2|$ of the broken line E_1HE_2 is independent from the position of H in the focal hyperbola.*

Proof. Using the notation of Example 1 in [3] the focal ellipse contains that points of the space which are of the form $(\sqrt{a^2-c^2}\cos u,\sqrt{b^2-c^2}\sin u,0)^T$ for $0\leq u<2\pi$ and the points of the two branches of the focal hyperbola have the coordinates $(\pm\sqrt{a^2-b^2}\cosh v, 0, -\sqrt{b^2-c^2}\sinh v)^T$. If H_i are the same branch of the hyperbola then we have

$$|H_1E|-|EH_2| = \sqrt{(\sqrt{a^2-c^2}\cos u \mp \sqrt{a^2-b^2}\cosh v_1)^2+(b^2-c^2)\sin^2 u+(b^2-c^2)\sinh^2 v_1}-$$

$$-\sqrt{(\sqrt{a^2-c^2}\cos u \pm \sqrt{a^2-b^2}\cosh v_2)^2+(b^2-c^2)\sin^2 u+(b^2-c^2)\sinh^2 v_2} =$$

$$= \sqrt{(a^2-b^2)\cos^2 u \mp 2\sqrt{a^2-c^2}\cos u\sqrt{a^2-b^2}\cosh v_1+(a^2-c^2)\cosh^2 v_1}-$$

$$-\sqrt{(a^2-b^2)\cos^2 u \mp 2\sqrt{a^2-c^2}\cos u\sqrt{a^2-b^2}\cosh v_2+(a^2-c^2)\cosh^2 v_2} =$$

$$= |\sqrt{a^2-b^2}\cos u \mp \sqrt{a^2-c^2}\cosh v_1| - |\sqrt{a^2-b^2}\cos u \mp \sqrt{a^2-c^2}\cosh v_2|.$$

Since $\sqrt{a^2-c^2}\cosh v > \sqrt{a^2-b^2}\cos u$ for all values u and v the terms containing the parameter u are vanishing showing the truth of the statement in this case. The other two statements of the lemma can be proved similarly. □

Lemma 3. *If P is any point of the space and l is a common transversal of the focal conics through P with points of intersection $E \in C_3 \cap l$ and $H \in C_2 \cap l$ for which E separates P and H (for which H separates P and E). Then for any point H_1 (E_1) on the same branch of the focal hyperbola (of the focal ellipse) as H is, the minimal length of the broken line PFH_1 (PFE_1) from P to H_1 (E_1) through a point F of the focal ellipse (of the same branch of the focal hyperbola as H) attend at the point E (H).*

If P separates the two points of intersection, then we have the inequality $|PE|-|EH_1| \leq |PF|-|FH_1|$.

Proof. $|PE|+|EH| \leq |PF|+|FH|$ for all points of C_3. $|HE|-|EH_1| = |HF|-|FH_1|$ implying that

$$|PE|+|EH_1| = |PE|+|EH|-(|EH|-|EH_1|) \leq |PF|+|FH|-(|FH|-|FH_1|) = |PF|+|FH_1|,$$

as we stated. The alternative statement in the brackets can be get with the same manner.

If P separates E and H then $|EH|-|PE| \geq |FH|-|FP|$ and again $|HE|-|EH_1| = |HF|-|FH_1|$, hence $|PE|-|EH_1| \leq |PF|-|FH_1|$ as we stated. □

Now we are ready to prove Staude's result on wire construction.

Theorem 3. *Let P be a point of the ellipsoid $\mathcal{E}$ given by the canonical form*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1.$$

Denote by $F_1 = (-\sqrt{a^2-c^2}, 0, 0)^T$ the left focus of the focal hyperbola of the confocal quadrics defined by the given ellipsoid and $G_2 = (\sqrt{a^2-b^2}, 0, 0)^T$ is the right focus of the focal ellipse of the same system. Then the sum of the shortest length of the broken line F_1HPEG_2 where H is a point of the focal hyperbola and E is a point of the focal ellipse is equal to $2a+\sqrt{a^2-c^2}-\sqrt{a^2-b^2}$ hence it is independent from the position of P on the ellipsoid.

Proof. To prove this statement use first Lemma 3 to calculate that distances which are needed for us. The common transversals of the focal conics have the parametric representation

$$\begin{aligned} \xi &= \varepsilon(\xi)t\sqrt{\frac{b^2c^2}{\mu\nu}} \\ \eta &= \varepsilon(\eta)t\sqrt{\frac{(b^2-\mu)(\mu-c^2)}{\mu(\nu-\mu)}} \\ \zeta &= \varepsilon(\zeta)t\sqrt{\frac{(\nu-b^2)(\nu-c^2)}{\nu(\nu-\mu)}} \end{aligned} \tag{2}$$

with respect to the dual coordinate system with origin P and having the normals of the confocals through P as axes. Here $a_1^1 = a$, $a_2^1 = b$, $a_3^1 = c$; $(a_1^2)^2 = a^2-\mu$, $(a_2^2)^2 = b^2-\mu$, $(a_3^2)^2 = c^2-\mu$; and $(a_1^3)^2 = a^2-\nu$, $(a_2^3)^2 = b^2-\nu$, $(a_3^3)^2 = c^2-\nu$ with $c^2 < \mu < b^2 < \nu < a^2$. One of the signs $\varepsilon(\xi), \varepsilon(\eta), \varepsilon(\zeta)$ can be chosen arbitrarily. The coordinate transformation which connects the variable ξ,η,ζ to the original variable x,y,z is

$$
\begin{aligned}
x-x' &= \left(\frac{p_0}{a^2}\xi+\frac{p_\mu}{a^2-\mu}\eta+\frac{p_\nu}{a^2-\nu}\zeta\right)x' \\
y-y' &= \left(\frac{p_0}{b^2}\xi+\frac{p_\mu}{b^2-\mu}\eta+\frac{p_\nu}{b^2-\nu}\zeta\right)y' \qquad (3)\\
z-z' &= \left(\frac{p_0}{c^2}\xi+\frac{p_\mu}{c^2-\mu}\eta+\frac{p_\nu}{c^2-\nu}\zeta\right)z',
\end{aligned}
$$

where the coordinates of the point P with respect to the original ($\{O,x,y,z\}$) system are x',y' and z', respectively and by (16) in [3]

$$
p_0:=\sqrt{\frac{a^2b^2c^2}{\mu\nu}},\ p_\mu=\sqrt{\frac{(a^2-\mu)(b^2-\mu)(\mu-c^2)}{\mu(\nu-\mu)}},\ p_\nu=\sqrt{\frac{(a^2-\nu)(\nu-b^2)(\nu-c^2)}{\nu(\nu-\mu)}}. \qquad (4)
$$

Hence the common edges with respect to the system $\{O,x,y,z\}$ have the following form:

$$
\begin{aligned}
x-x' &= t\left(\varepsilon(\xi)\frac{b^2c^2}{a\mu\nu}-\varepsilon(\eta)\frac{(b^2-\mu)(c^2-\mu)}{\sqrt{a^2-\mu}\mu(\nu-\mu)}+\varepsilon(\zeta)\frac{(b^2-\nu)(c^2-\nu)}{\sqrt{a^2-\nu}\nu(\nu-\mu)}\right)x' \\
y-y' &= t\left(\varepsilon(\xi)\frac{ac^2}{\mu\nu}-\varepsilon(\eta)\frac{\sqrt{a^2-\mu}(c^2-\mu)}{\mu(\nu-\mu)}+\varepsilon(\zeta)\frac{\sqrt{a^2-\nu}(c^2-\nu)}{\nu(\nu-\mu)}\right)y' \qquad (5)\\
z-z' &= t\left(\varepsilon(\xi)\frac{ab^2}{\mu\nu}-\varepsilon(\eta)\frac{\sqrt{a^2-\mu}(b^2-\mu)}{\mu(\nu-\mu)}+\varepsilon(\zeta)\frac{\sqrt{a^2-\nu}(b^2-\nu)}{\nu(\nu-\mu)}\right)z'.
\end{aligned}
$$

The focal ellipse lying on the plane $z=0$ implying that the corresponding possible parameters are

$$
t=\frac{-1}{\left(\varepsilon(\xi)\frac{ab^2}{\mu\nu}-\varepsilon(\eta)\frac{\sqrt{a^2-\mu}(b^2-\mu)}{\mu(\nu-\mu)}+\varepsilon(\zeta)\frac{\sqrt{a^2-\nu}(b^2-\nu)}{\nu(\nu-\mu)}\right)}=
$$

$$
=\frac{\mu\nu(\nu-\mu)}{-\varepsilon(\xi)ab^2(\nu-\mu)+\varepsilon(\eta)\sqrt{a^2-\mu}(b^2-\mu)\nu-\varepsilon(\zeta)\sqrt{a^2-\nu}(b^2-\nu)\mu},
$$

which square gives the squared distance of P to the corresponding point E of the transversal by equation (26) in [3]. From this we get

$$
x=\frac{(a^2-c^2)\left(-\varepsilon(\xi)\frac{b^2(\nu-\mu)}{a}+\varepsilon(\eta)\frac{\nu(b^2-\mu)}{\sqrt{a^2-\mu}}-\varepsilon(\zeta)\frac{\mu(b^2-\nu)}{\sqrt{a^2-\nu}}\right)}{-\varepsilon(\xi)ab^2(\nu-\mu)+\varepsilon(\eta)\sqrt{a^2-\mu}(b^2-\mu)\nu-\varepsilon(\zeta)\sqrt{a^2-\nu}(b^2-\nu)\mu}x'=
$$

$$
=\frac{(a^2-c^2)\left(-\varepsilon(\xi)\frac{b^2(\nu-\mu)}{a}+\varepsilon(\eta)\frac{\nu(b^2-\mu)}{\sqrt{a^2-\mu}}-\varepsilon(\zeta)\frac{\mu(b^2-\nu)}{\sqrt{a^2-\nu}}\right)}{-\varepsilon(\xi)ab^2(\nu-\mu)+\varepsilon(\eta)\sqrt{a^2-\mu}(b^2-\mu)\nu-\varepsilon(\zeta)\sqrt{a^2-\nu}(b^2-\nu)\mu}\varepsilon(x')\frac{a\sqrt{a^2-\mu}\sqrt{a^2-\nu}}{\sqrt{a^2-b^2}\sqrt{a^2-c^2}}=
$$

$$
=\varepsilon(x')\frac{\sqrt{a^2-c^2}}{\sqrt{a^2-b^2}}\frac{\left(-\varepsilon(\xi)b^2(\nu-\mu)\sqrt{a^2-\mu}\sqrt{a^2-\nu}+\varepsilon(\eta)\nu(b^2-\mu)a\sqrt{a^2-\nu}-\varepsilon(\zeta)\mu(b^2-\nu)a\sqrt{a^2-\mu}\right)}{\left(-\varepsilon(\xi)ab^2(\nu-\mu)+\varepsilon(\eta)\sqrt{a^2-\mu}(b^2-\mu)\nu-\varepsilon(\zeta)\sqrt{a^2-\nu}(b^2-\nu)\mu\right)},
$$

where $\varepsilon(x')$ is the sign of x' and we used here equation (12), too. For brevity denote by the long fraction by B/A meaning that

$$
x:=\varepsilon(x')\frac{\sqrt{a^2-c^2}}{\sqrt{a^2-b^2}}\frac{B}{A}.
$$

Applying in the plane $z=0$ the equation (12) for the focal ellipse we get that there is a constant e for which $b^2\le e^2\le a^2$ and

$$
x=\varepsilon((C_3)_x)\frac{\sqrt{a^2-c^2}\sqrt{a^2-e^2}}{\sqrt{a^2-b^2}}\text{ and }y=\varepsilon((C_3)_y)\frac{\sqrt{b^2-c^2}\sqrt{e^2-b^2}}{\sqrt{a^2-b^2}}
$$

where $\varepsilon((C_3)_x)$ is the sign of the coordinate x. This implies the equality:

$$
\varepsilon(x')\varepsilon((C_3)_x)\sqrt{a^2-e^2}=\frac{B}{A}.
$$

On the other hand we have the identity

$$\varepsilon(\xi)\varepsilon(\eta)\varepsilon(\zeta)B+(\varepsilon(\xi)a+\varepsilon(\eta)\sqrt{a^2-\mu}+\varepsilon(\zeta)\sqrt{a^2-\nu})A=$$
$$=-\varepsilon(\eta)\varepsilon(\zeta)b^2(\nu-\mu)\sqrt{a^2-\mu}\sqrt{a^2-\nu}+\varepsilon(\xi)\varepsilon(\zeta)\nu(b^2-\mu)a\sqrt{a^2-\nu}-\varepsilon(\xi)\varepsilon(\eta)\mu(b^2-\nu)a\sqrt{a^2-\mu}+$$
$$+(\varepsilon(\xi)a+\varepsilon(\eta)\sqrt{a^2-\mu}+\varepsilon(\zeta)\sqrt{a^2-\nu})(-\varepsilon(\xi)ab^2(\nu-\mu)+\varepsilon(\eta)\sqrt{a^2-\mu}(b^2-\mu)\nu-\varepsilon(\zeta)\sqrt{a^2-\nu}(b^2-\nu)\mu)=$$
$$=\varepsilon(\eta)\varepsilon(\zeta)\left(-b^2(\nu-\mu)-(b^2-\nu)\mu+(b^2-\mu)\nu\right)\sqrt{a^2-\mu}\sqrt{a^2-\nu}+$$
$$+\varepsilon(\xi)\varepsilon(\zeta)\left(\nu(b^2-\mu)-(b^2-\nu)\mu-b^2(\nu-\mu)\right)a\sqrt{a^2-\nu}+$$
$$+\varepsilon(\xi)\varepsilon(\eta)\left(-\mu(b^2-\nu)-b^2(\nu-\mu)+(b^2-\mu)\nu\right)a\sqrt{a^2-\mu}+$$
$$-a^2b^2(\nu-\mu)+(a^2-\mu)(b^2-\mu)\nu-(a^2-\nu)(b^2-\nu)\mu=\mu^2\nu-\nu^2\mu=-\mu\nu(\nu-\mu),$$

implying the result:

$$t=\frac{\mu\nu(\nu-\mu)}{A}=-\varepsilon(\xi)a-\varepsilon(\eta)\sqrt{a^2-\mu}-\varepsilon(\zeta)\sqrt{a^2-\nu}-\varepsilon(\xi)\varepsilon(\eta)\varepsilon(\zeta)\varepsilon(x')\varepsilon((C_3)_x)\sqrt{a^2-e^2}. \tag{6}$$

Similarly the focal hyperbola is lying on the plane $y=0$. Hence we have in this case that

$$\tau=\frac{\mu\nu(\nu-\mu)}{-\varepsilon(\xi)ac^2(\nu-\mu)+\varepsilon(\eta)\sqrt{a^2-\mu}(c^2-\mu)\nu-\varepsilon(\zeta)\sqrt{a^2-\nu}(c^2-\nu)\mu},$$

and also

$$\bar{x}=\varepsilon(x')\frac{\sqrt{a^2-b^2}}{\sqrt{a^2-c^2}}\cdot$$
$$\cdot\frac{-\varepsilon(\xi)c^2(\nu-\mu)\sqrt{a^2-\mu}\sqrt{a^2-\nu}+\varepsilon(\eta)\nu(c^2-\mu)a\sqrt{a^2-\nu}-\varepsilon(\zeta)\mu(c^2-\nu)a\sqrt{a^2-\mu}}{-\varepsilon(\xi)ac^2(\nu-\mu)+\varepsilon(\eta)\sqrt{a^2-\mu}(c^2-\mu)\nu-\varepsilon(\zeta)\sqrt{a^2-\nu}(c^2-\nu)\mu}=$$
$$=\varepsilon(x')\frac{\sqrt{a^2-b^2}}{\sqrt{a^2-c^2}}\frac{D}{C}$$

For the coordinates of the focal hyperbola by (12) hold that

$$\bar{x}=\varepsilon((C_2)_x)\frac{\sqrt{a^2-b^2}\sqrt{a^2-f^2}}{\sqrt{a^2-c^2}},\ y=0 \text{ and } z=\varepsilon((C_2)_z)\frac{\sqrt{b^2-c^2}\sqrt{f^2-c^2}}{\sqrt{a^2-c^2}},$$

hence we have

$$\varepsilon(x')\varepsilon((C_2)_{\bar{x}})\sqrt{a^2-f^2}=\frac{D}{C}.$$

Clearly the identity

$$\varepsilon(\xi)\varepsilon(\eta)\varepsilon(\zeta)D+(\varepsilon(\xi)a+\varepsilon(\eta)\sqrt{a^2-\mu}+\varepsilon(\zeta)\sqrt{a^2-\nu})C=-\mu\nu(\nu-\mu)$$

is also hold, and thus

$$\tau=\frac{\mu\nu(\nu-\mu)}{C}=-\varepsilon(\xi)a-\varepsilon(\eta)\sqrt{a^2-\mu}-\varepsilon(\zeta)\sqrt{a^2-\nu}-\varepsilon(\xi)\varepsilon(\eta)\varepsilon(\zeta)\varepsilon(x')\varepsilon((C_2)_x)\sqrt{a^2-f^2}. \tag{7}$$

For a given position of the point P choose that four combinations for the eight possible ones that the four intersection points on the focal ellipse correspond to positive parameter values, respectively. It is easy to see[1] that in this case we have to choose the following combination of signs:

	$\varepsilon(\xi)$	$\varepsilon(\eta)$	$\varepsilon(\zeta)$
r_1	-1	1	1
r_2	-1	1	-1
r_3	-1	-1	-1
r_4	-1	-1	1

,

The corresponding common transversals we call the focal radiuses r_i of the point P to the focal conics.

In this case, we can compare the positions of the points E and H on a focal radius by the parameter values t and τ. If $\tau>t>0$ then the point E separates the point P and H and if $t>\tau$ then either H separates P and E or P separates E and H, respectively. Consider the difference

$$\frac{1}{\tau}-\frac{1}{t}=-\frac{b^2-c^2}{\mu\nu(\nu-\mu)}\left(-\varepsilon(\xi)a(\nu-\mu)+\varepsilon(\eta)\sqrt{a^2-\mu}\nu-\varepsilon(\zeta)\sqrt{a^2-\nu}\mu\right)=$$

[1]We have to prove that the equality $ab^2(\nu-\mu)>\sqrt{a^2-\mu}(b^2-\mu)\nu+\sqrt{a^2-\nu}(b^2-\nu)\mu$ fulfill if $c^2<\mu<b^2<\nu<a^2$ hold for the corresponding parameters.

$$= \frac{b^2-c^2}{\mu\nu(\nu-\mu)}(\varepsilon(\eta)\sqrt{a^2-\mu}-\varepsilon(\zeta)\sqrt{a^2-\nu})(\varepsilon(\zeta)\sqrt{a^2-\nu}-\varepsilon(\xi)a)(\varepsilon(\xi)a-\varepsilon(\eta)\sqrt{a^2-\mu}).$$

This value is positive for r_3 and r_4 and negative for r_1 and r_2, respectively. This implies that for $i=1,2$ we have two possibilities. If $\tau > o$ then the focal radius r_i first intersect the focal ellipse and than the focal hyperbola and if $\tau < 0$ then P separates the two intersection points to each other. In the case of $i=3,4$ τ is always positive the two points of intersections are on the same radius and the point of the focal hyperbola separates the point P and the point of the focal ellipse. For $i=3$ the numerator D of r_3 is negative and the same value for r_4 is positive. Since the denominator C is positive in both of these cases, the corresponding points of the focal hyperbola have opposite half-spaces with respect to the yz-plane. Hence in the case of r_3 $\varepsilon((C_2)_x)=-\varepsilon(x')$ and in the case of r_4 $\varepsilon((C_2)_x)=\varepsilon(x')$, respectively. Since for r_1 D is positive and for r_2 D is negative then the sign dependent from the relative position of the point P and the points of the focal hyperbola. For r_1 with respect to the fact that τ is positive or negative, P has to the same or the opposite half-space as the point of the focal hyperbola. Conversely, for r_2 the point of the focal hyperbola lies in the opposite or same half-space as P with respect to the fact that *tau* positive or negative, respectively. By formula we have that in the case of r_1 $\varepsilon((C_2)_x)=\varepsilon(x')\tau$ and in the case of r_2 $\varepsilon((C_2)_x)=-\varepsilon(x')\tau$.

Hence the broken line PEG_2 can be realised with focal radii r_1 or r_2 meaning that either

$$|PE| = t = a-\sqrt{a^2-\mu}-\sqrt{a^2-\nu}+\varepsilon(x')\varepsilon((C_3)_x)\sqrt{a^2-e^2} \text{ in the case of } r_1$$

or

$$|PE| = t = a-\sqrt{a^2-\mu}+\sqrt{a^2-\nu}-\varepsilon(x')\varepsilon((C_3)_x)\sqrt{a^2-e^2} \text{ in the case of } r_2,$$

respectively. For the distance of E and G_2 we have

$$(EG_2)^2 = \left(\rho\frac{\sqrt{a^2-c^2}\sqrt{a^2-e^2}}{\sqrt{a^2-b^2}}-\sqrt{a^2-b^2}\right)^2+\left(\rho\frac{\sqrt{b^2-c^2}\sqrt{e^2-b^2}}{\sqrt{a^2-b^2}}\right)^2=$$

$$=(-e^2)+(a^2-b^2)+(a^2+b^2)+(-c^2)-2\rho\sqrt{(a^2-c^2)(a^2-e^2)}=$$

$$=\left(\sqrt{a^2-c^2}-\rho\sqrt{a^2-e^2}\right)^2,$$

where ρ is $+1$ if E and G_2 are in the same half-space with respect to the plane yz and -1 in the other case. From this we get that with respect to the fact that G_2 and P are the same or opposite half-spaces we get the optimal polygonal lengths:

$$|PE|+|EG_2| = a-\sqrt{a^2-\mu}-\varepsilon\sqrt{a^2-\nu}+\sqrt{a^2-c^2},$$

where ε is positive or negative with respect to that $\varepsilon(x')$ positive or negative, respectively.

Similarly, the broken line PHF_1 can be realised in the cases when we have either consider r_3 with

$$|PH| = \tau = a+\sqrt{a^2-\mu}+\sqrt{a^2-\nu}-\sqrt{a^2-f^2}$$

or r_4 with

$$|PH| = \tau = a+\sqrt{a^2-\mu}-\sqrt{a^2-\nu}-\sqrt{a^2-f^2},$$

respectively.

Since we have that

$$|HF_1| = \sqrt{\left(\varepsilon((C_2)_{\bar{x}})\frac{\sqrt{a^2-b^2}\sqrt{a^2-f^2}}{\sqrt{a^2-c^2}}+\sqrt{a^2-c^2}\right)^2+(\varepsilon((C_2)_z)\frac{\sqrt{b^2-c^2}\sqrt{f^2-b^2}}{\sqrt{a^2-c^2}})^2}=$$

$$=\sqrt{a^2-b^2}+\varepsilon((C_2)_{\bar{x}})\sqrt{a^2-f^2},$$

and we get for the polygonal length of PHF_1

$$|PH|+|HF_1| = a+\sqrt{a^2-\mu}+\varepsilon\sqrt{a^2-\nu}+\sqrt{a^2-b^2}$$

if H and F_1 corresponding to distinct branches of the focal hyperbola and ε is positive or negative with respect to that P and F_1 are on the same or opposite halfspaces of the yz plane. If H and F_1 corresponding to the same branch of the focal hyperbola we have

$$|PH|+|HF_1| = a+\sqrt{a^2-\mu}-\varepsilon\sqrt{a^2-\nu}-\sqrt{a^2-b^2},$$

where again $\varepsilon=1$ if and only if P and F_1 are on the same half-space of the yz plane.

Comparing the above results with the assumption of the statement we get that the optimal polygonal length is

$$|PE| + |EG_2| + |PH| + |HF_1| = 2a + \sqrt{a^2 - c^2} - \sqrt{a^2 - b^2},$$

as we stated. □

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Department of Geometry, Budapest University of Technology and Economics, H-1521 Budapest, Hungary

E-mail address: ghorvath@math.bme.hu

A unit vector in the Euclidean space â„2. v = {3, 7}; u=1Normvv.Â The shape of a surface M in Euclidean space R3 is described infinitesimally by a linear operator S defined on each of the tangent planes of M. This chapter justify this infinitesimal measurements by proving that two surfaces with â€œthe sameâ€ shape operators are, in fact, congruent. The algebraic invariants of its shape operators thus have geometric meaning for the surface M. The chapter highlights efficient ways to compute these invariants, and test them on a number of geometrically interesting surfaces.