Infinite-Dimensional Lie Groups

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Introduction

It is frequently possible to describe the symmetries of geometric (or other mathematical) objects by finitely many real parameters. This leads to the notion of a finite-dimensional Lie group, which locally looks like euclidean space and whose group operations are smooth (i.e., C^∞-) maps.

Example 1: We can think of the group $\mathrm{GL}_n(\mathbb{R})$ of all invertible $(n \times n)$-matrices. Since

$$\mathrm{GL}_n(\mathbb{R}) \;=\; \det{}^{-1}(\mathbb{R}^\times)$$

where $\det\colon M_n(\mathbb{R}) \to \mathbb{R}$ is continuous, $\mathrm{GL}_n(\mathbb{R})$ is an open subset of the space $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ of $(n \times n)$-matrices, and the matrices $A = (a_{ij}) \in \mathrm{GL}_n(\mathbb{R})$ are parametrized by the n^2 parameters a_{ij}, where $i, j \in \{1, \ldots, n\}$. The group multiplication on $\mathrm{GL}_n(\mathbb{R})$ is a restriction of the matrix multiplication

$$M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to M_n(\mathbb{R})\,, \qquad (A, B) \mapsto AB\,,$$

which is bilinear and hence smooth (the smoothness is also apparent from the explicit formula $(AB)_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$ for the matrix entries of AB). Using Cramer's rule to calculate A^{-1}, we see that the matrix entries of A^{-1} are rational functions in the matrix entries a_{ij} of A and therefore smooth functions on the open subset $\mathrm{GL}_n(\mathbb{R}) \subseteq M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ as well.

In other situations, finitely many real parameters do not suffice, but it is still possible to describe the elements of a symmetry group by means of a parameter in a *topological vector space*.

Example 2. Given a real Banach space E, let $\mathcal{L}(E)$ be the Banach space of all continuous linear operators $A\colon E \to E$, equipped with the operator norm, and $\mathrm{GL}(E) := \mathcal{L}(E)^\times$ be the group of all invertible operators. As we shall verify later, $\mathrm{GL}(E)$ is open in $\mathcal{L}(E)$. Furthermore–although we cannot use Cramer's rule anymore now—it it still true that the inversion map

$$\iota\colon \mathrm{GL}(E) \to \mathrm{GL}(E)\,, \qquad \iota(A) \;:=\; A^{-1} \tag{0.1}$$

is smooth in a suitable sense, as well as the composition map

$$\mathrm{GL}(E) \times \mathrm{GL}(E) \to \mathrm{GL}(E)\,, \qquad (A, B) \mapsto A \circ B\,.$$

Of course, we need to define what a "smooth map" between open subsets of topological vector spaces is, before we can speak of (and prove) smoothness of mappings like ι in (0.1), which is a mapping on the open set $\mathrm{GL}(E) \subseteq \mathcal{L}(E)$. The lecture course will therefore start with an introduction to differential calculus in topological vector spaces. As this suffices for all applications, we shall restrict our attention to differential calculus in *locally convex* topological vector spaces, which is easier than the general case. Note that, in Example 2, we do not simply have finitely many real parameters (coordinates), as the a_{ij} in the first example, but a single parameter in a topological vector space (namely in $\mathcal{L}(E)$).[1] This is typical for infinite-dimensional analysis, and makes it necessary to formulate differential calculus in a *coordinate free* way.

Of course, the preceding examples over-simplify the case, because the groups are mere open subsets of a topological vector space there. In general, the groups we are interested in will only *locally* look like an open subset of a locally convex space, they will be *smooth manifolds* modelled on locally convex spaces (in a sense to be defined later). The eventual definition of our objects of study will be as follows:

Definition. A *Lie group* is a group, G, equipped with a smooth manifold structure modelled on a locally convex space E, which turns the inversion map

$$\iota\colon G \to G\,, \quad x \mapsto x^{-1}$$

and the group multiplication

$$m\colon G \times G \to G\,, \quad (x, y) \mapsto xy$$

into smooth mappings.

This definition makes sense if E is a real locally convex space, in which case G is a *real* Lie group (this is the most important case). But we can also apply it if E is a complex locally convex space, using a suitable concept of smooth maps based on complex differentiability. Thus, to enable the discussion of *complex* Lie groups, we also need to broach on infinite-dimensional (and multi-dimensional) analogues of the holomorphic functions familiar from undergraduate single-variable complex function theory.

Let us illustrate the occurrence of manifolds by an example.

Example. $\mathrm{SL}_2(\mathbb{R}) := \{A \in \mathrm{GL}_2(\mathbb{R})\colon \det(A) = 1\}$ is a subgroup of $\mathrm{GL}_2(\mathbb{R})$, called the *special linear group*. The set $\mathrm{SL}_2(\mathbb{R})$ is a 3-dimensional submanifold of $M_2(\mathbb{R}) \cong \mathbb{R}^4$, because it is the set of zeros of the map

$$M_2(\mathbb{R}) \to \mathbb{R}\,, \quad A \mapsto \det(A) - 1$$

corresponding to the smooth map

[1] We parametrize $A \in \mathrm{GL}(E) \subseteq \mathcal{L}(E)$ by itself.

$$f : \mathbb{R}^4 \to \mathbb{R}, \quad f(a,b,c,d) := ad - bc - 1,$$

whose gradient

$$(\operatorname{grad} f)(a,b,c,d) = (d, -c, -b, a)$$

is non-zero whenever $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$.

The subject matter of this course can be subdivided into for main areas:

Calculus in locally convex spaces. An introduction to the calculus of smooth and analytic mappings in locally convex spaces will be given.

Basic theory of infinite-dimensional Lie groups. Basic definitions and facts concerning Lie groups and manifolds modelled on locally convex spaces will be provided.

Lie group constructions. The construction of the main examples of infinite-dimensional Lie groups will be explained in detail. In particular, we shall discuss

- *Linear Lie groups*, like $\mathrm{GL}(E)$ encountered above or the group $U(\mathcal{H})$ of all unitary operators on a complex Hilbert space $\mathcal{H}$;
- *Mapping groups*, like the group $C^r(K,G)$ of C^r-maps from a compact smooth manifold K to a Lie group G (using pointwise multiplication and inversion as the group operations);
- *Diffeomorphism groups*, notably the group $\mathrm{Diff}(\mathbb{R}^n)$ of smooth diffeomorphisms of $\mathbb{R}^n$, with composition of maps as the multiplication.

Specific tools from differential calculus. Specific, more specialized tools of infinite-dimensional calculus will be introduced to simplify the Lie group constructions. In particular, we shall see how differential calculus on *metrizable* locally convex spaces can be simplified.

Of course, the usefulness of calculus in locally convex spaces is not restricted to Lie theory; it permits applications also in other branches of mathematics. Furthermore, the occupation with infinite-dimensional, coordinate-free analysis will give the reader a new perspective on finite-dimensional calculus, and a deeper insight.

In this connection, it is also worth noting that we shall not only discuss continuously differentiable mappings on open subsets of locally convex spaces, but more generally mappings on suitable sets with non-empty interior, thus enabling us to consider mappings on $[0,1]$, $[0,1]^2$ and similar sets in our development of Lie theory. This is interesting also in the finite-dimensional case, because undergraduate calculus courses usually focus on the case of open domains, although the case of non-open domains is needed in the applications (for example, to define manifolds with boundary).[2] Furthermore, as already

[2] Which only occur implicitly in the usual Darmstadt curriculum, in the guise of the "compact sets with smooth boundary" in the discussion of Gauß' integral theorem in the analysis course on multiple integration/vector analysis.

mentioned, we shall discuss real and complex analytic mappings, the basic theory of which is useful also in the finite-dimensional case.

Let us remark in closing that finite-dimensional Lie theory (in local form) was initiated by the Norwegian mathematician Marius Sophus Lie (1842–1899). The theory was then brought to its current form in the 20th century and is still an active area of research which involves techniques from (and is useful for) many branches of mathematics, like differential geometry, harmonic analysis, and representation theory. In parallel to the development of finite-dimensional Lie theory, analogous theories have been devised for infinite-dimensional groups of ever increasing generality. This started with the consideration of groups of operators and the subsequent development of the basic (local) theory of Banach-Lie groups (i.e., Lie groups modelled on Banach spaces) in the 1930s. In modern (global) form, the theory of Banach-Lie groups was spelled out by the french collective N. Bourbaki[3] in the early 1970s (as part of their "Elements of Mathematics," an axiomatic, self-contained exposition of the foundations of mathematics). In the 1970s, Fréchet-Lie groups entered the scene (like mapping groups $C^\infty(K, G)$ with G a finite-dimensional Lie group, and the diffeomorphism groups of compact smooth manifolds). Because many familiar facts from differential calculus (like the inverse function theorem) carry over without problems to Banach spaces, but become false for Fréchet spaces, the step from Banach-Lie groups to Fréchet-Lie groups was not taken without difficulty, and various problems (including psychological barriers) had to be overcome. From the 1980s, then, infinite-dimensional Lie groups started to be considered (almost) in the generality considered in the present text, modelled on arbitrary sequentially complete (or slightly more general) locally convex spaces.

We shall follow the approach to infinite-dimensional Lie groups popularized by John Milnor (a Fields medalist) in the 1980s, based on a framework of infinite-dimensional calculus known as "Keller's C_c^∞-theory." Whenever this is useful for our purposes, we shall weave in ideas from the so-called "Convenenient Differential Calculus," an equally important, inequivalent setting of analysis developed by Frölicher, Kriegl and Michor, which is based on a weaker concept of smooth mappings (that need not even be continuous).

Because the objects, problems and techniques of infinite-dimensional Lie theory differ substantially from the finite-dimensional case, no knowledge of finite-dimensional Lie theory is necessary for an understanding of the present text, which essentially is a treatise on aspects of non-linear functional analysis. While we shall focus entirely on the theory and main examples, let us mention at least that mapping groups (and related infinite-dimensional Lie groups like gauge groups of principal bundles) are encountered in theoretical physics (in quantum mechanics), whereas diffeomorphism groups are useful

[3] The name "N. Bourbaki" served as the collective's pseudonym.

tools in hydrodynamics.

Prerequisites. The reader should have some familiarity with topological spaces (beyond metric spaces) and should know the basic facts of functional analysis in normed spaces. However, the actual framework for our studies will not be normed spaces, but general locally convex topological vector spaces. Therefore, a self-contained introduction to the basic theory of locally convex spaces (with full proofs) is provided in an appendix. It is essential for an understanding of the text to know the basic facts compiled there (albeit not the proofs). Previous occupation with finite-dimensional Lie groups or Lie algebras is not required. Another appendix devoted to basic topological facts will be provided later, for the reader's convenience.

We announce new results on the cohomology groups of certain infinite dimensional Lie algebras (Theorems 3.2 and 3.4, below). These Lie algebras are subalgebras of algebras introduced by Moody and Kac (see refs. 1 and 2). We also obtain an empirical relation between these cohomology groups and the cohomology of loop spaces of compact groups (see Corollary 3.6, below). Finally, we derive Macdonald's identities for the powers of the Dedekind n-function (see ref. 3), by applying the Euler-Poincare principle and our cohomology results. In ref. 4, Kac obtained a generalization of Macdonald&apos A representation of a Lie group (cf. Representation of a topological group) in an infinite-dimensional vector space. The theory of representations of Lie groups is part of the general theory of representations of topological groups. The specific features of Lie groups make it possible to employ analytical tools in this theory (in particular, infinitesimal methods), and also to considerably enlarge the class of "natural" group algebras (function algebras with respect to convolution, cf. Group algebra)