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Maps between classifying spaces of the unitary groups

Doctoral dissertation summary

The dissertation is devoted to a classical problem of Homotopy Theory, namely homotopy classification of maps between classifying spaces of compact Lie groups. The main result is a construction of new exotic maps between classifying spaces of the unitary groups $BU_n \to BU_m$. If $n > 18$ and $m \leqslant t(n) := \frac{1}{2}n(n-1)(n+2)$ we obtain certain classification of such maps.

Let G be a compact connected Lie group. A (homotopy class) of a map $f\colon BG \to BU_m$ will be called a homotopy representation of the group G. For a prime p, a p-toral group is a group whose identity component is a torus and group of connected components is a finite p-group. According to the Dwyer-Zabrodsky-Notbohm Theorem [11, 26] a homotopy representation f restricted to any p-toral subgroup $P \subseteq G$ is induced by a group homomorphism $\rho^f_P\colon P \to U_m$, i.e. $f|_{BP} \sim B\rho^f_P$. In particular, if $P = \mathrm{T}_G \subseteq G$ is a maximal torus we obtain a representation $\rho^f_{\mathrm{T}_G}\colon \mathrm{T}_G \to U_m$ which is invariant (up to an isomorphism) under the action of the Weyl group $W_G := N_G(\mathrm{T}_G)/\mathrm{T}_G$ on T_G. Thus the homotopy representation f defines a W_G-invariant element in $R(\mathrm{T}_G)$ – the representation ring of the torus. Since the restriction homomorphism $\mathrm{res}^G_{\mathrm{T}_G}\colon R(G) \to R(T)^{W_G}$ is an isomorphism [5], we can associate to every homotopy representation f its character $\rho^f \in R(G)$. The construction obviously generalizes the construction of a character of a linear representation.

The main question is: what characters $\mu \in R(G)$ are homotopy characters? The Dwyer-Zabrodsky-Notbohm Theorem implies that for any p-toral subgroup P restriction $\mu|_P \in R^+(P)$ must be a genuine representation of P. Characters of G having such property we will call p-characters, and those which are p-characters for every prime p will be called $\mathcal{P}$-characters. Thus the first step in classification of maps $BU_n \to BU_m$ is a characterisation of $\mathcal{P}$-characters of U_n. This purely algebraic question is considered in Chapters 1-4.

We begin Chapter 1 with recalling basic definitions in representation theory and then define p-characters and prove their elementary properties. In Section 1.5 we define a slant-product in representation ring of a product of two groups which, in subsequent sections, is used for reduction from larger to smaller subgroups. In some special cases the operation is called a reduction of a character.

In Chapter 2 we apply reduction of characters to formulate criteria when a virtual character of a unitary group is a p-character. For that we need a careful description of the maximal p-toral subgroup of U_n as an iterated wreath product of one-dimensional torus and cyclic groups (Sec. 2.2) and study its representations (Sec. 2.3). In Sec. 2.4 the main characterization theorems of p-characters of the unitary groups [**??**], [**??**] are proved. Detailed study of the case of U_p is carried on in Sec. 2.5 resulting in a simple characterization of p-characters of U_p **??**. In Sec. 2.6 we describe a group endomorphism of the maximal p-toral subgroup $N^n_p \subseteq U_n$ which defines the Adams operation $\Psi^k\colon R(N^n_p) \to R(N^n_p)$. In general, effect of the k-th Adams operation on a character we call its k-twisting.

In Chapter 3 we construct families of $\mathcal{P}$-characters of U_n. In Sect. 3.1 we describe some representations of U_n which are used for construction of p-characters of U_n. In Section 3.2 we list candidates for $\mathcal{P}$-characters and check which of them actually are $\mathcal{P}$-characters. In last

Section 3.3 we describe examples showing that a decomposition of p-characters into sum of the indecomposable p-characters is not unique.

In Chapter 4 we show that for unitary group U_n such that $n > 18$, $\mathcal{P}$-characters of dimension $\leqslant t(n)$ are exactly the ones constructed in Chapter 3. A proof relies on a careful analysis of dimensions of the symmetrized weights of the torus which can occur in decompositions of $\mathcal{P}$-characters restricted to the maximal torus. Proof of the main algebraic result is presented in Section 4.4.

The last Chapter 5 is devoted to topological application of the algebraic result which occupies Chapters 1-4. The main result says that all $\mathcal{P}$-characters listed in Section 4.4 are indeed homotopy characters. The key idea is a splitting property of characters recalled in Sec. 5.2. We say that a (homotopy) character $\nu \in R(G)$ has a splitting property if any character χ such that $\chi + \mu$ is a homotopy character is also a homotopy character. Results of [19], and a recent paper [21] imply that the trivial character and characters of the Adams operations have splitting property. We are lucky since the results, combined with other results of [15], suffice to prove that all our $\mathcal{P}$-characters are indeed homotopy characters of U_n.

All the groups considered in this dissertation are compact Lie; all subgroups considered are closed. We denote by Grp the category of compact Lie groups and their homomorphisms.

Literatura

[1] J.F. Adams, *Lectures on Lie groups*, Benjamin (1969).

[2] J.F. Adams, Z. Mahmud, *Maps between Classifying Spaces*, Inventiones math. 35 (1976), 1-41.

[3] A.K. Bousfield, *The localization of spaces with respect to homology*, Topology 14 (1975), 133-150.

[4] A.K. Bousfield, D.M. Kan, *Homotopy Limits, Completions and Localizations*, Lecture Notes in Math. 304, Springer, New York, 1972.

[5] T. Bröcker, T. tom Dieck, *Representations of Compact Lie Groups*, Springer-Verlag, 1985.

[6] G. Carlsson, *Equivariant stable homotopy and Sullivan's conjecture*, Invent. math. 103 (1991), 497-525.

[7] W. Dwyer, *Classifying Spaces and Homology Decompositions*, available online http://www3.nd.edu/ wgd/Dvi/Classifying.Spaces.Homology.Decompositions.pdf

[8] W. Dwyer, H. Miller, J. Neisendorfer, *Fibrewise completion and unstable Adams spectral sequences*, Israel J. Math. 66 (1989), 160-178.

[9] W. Dwyer, C. Wilkerson, *Homotopy fixed point methods for Lie groups and finite loop spaces*, Ann. Math. 139 (1994), 395-442.

[10] W. Dwyer, C. Wilkerson, *Maps of BZ/pZ to BG*, Algebraic Topology - Rational Homotopy, Lecture Notes in Math. 1318 (1988), 92-98.

[11] W. Dwyer, A. Zabrodsky, *Maps Between Classifying Spaces*, Algebraic Topology, Barcelona 1986, Lecture Notes in Math. 1298 (1987), 106-119.

[12] J. Grodal, *Higher limits via subgroup complexes*, Ann Math 155 (2002), 405-457.

[13] W. Hurewicz, *Beiträge zur Topologie der Deformationen IV: Asphärische Räume*, Proc. Akad. Wetensch. Amsterdam 39 (1936), 215-224.

[14] K. Ishiguro, *Unstable Adams operations on classifying spaces*, Math. Proc. Camb. Phil. Soc. 102 (1987), 71-75.

[15] S. Jackowski, J. McClure, B. Oliver, *Homotopy classification of self-maps of BG via G-actions*, Ann. Math. 135 (1992), 183-226.

[16] S. Jackowski, J. McClure, B. Oliver, *Homotopy theory of classifying spaces of compact lie groups*, Algebraic Topology and its Applications, Springer-Verlag (1994), 81-124.

[17] S. Jackowski, J. McClure, B. Oliver, *Maps between classifying spaces revisited*, Contemp. Math. 181 (1993), 263-298.

[18] S. Jackowski, J. McClure, B. Oliver, *Self-homotopy equivalences of classifying spaces of compact connected Lie groups*, Fund. Math. 147 (1995), 99-126.

[19] S. Jackowski, B. Oliver, *Vector bundles over classifying spaces of compact Lie groups*, Acta Math. 176/1 (1996), 109-143.

[20] J. Lannes, *Sur la cohomologie modulo p des p-groupes abéliens élémentaires*, Homotopy theory, Proc. of the Durham symposium 1985, Cambridge Univ. Press (1987), 97-116.

[21] W. Lubawski, K. Ziemiański, *Homotopy representations of the unitary groups*, arXiv:1406.7467.

[22] M. Mather, *Pull-backs in homotopy theory*, Can. J. Math. XXVIII (1976), 225-263.

[23] J.P. May, *Simplicial objects in algebraic topology*, The University of Chicago Press (1967).

[24] J.P. May, *The Geometry of Iterated Loop Spaces*, Lecture Notes in Math. 271 (1972).

[25] J. Milnor, *Construction of universal bundles II*, Ann Math 63 (1956), 430-436.

[26] D. Notbohm, *Classifying spaces for compact Lie groups*, Handbook of Algebraic Topology, North-Holland, 1995, 1049-1094.

[27] B. Oliver, *Higher limits via Steinberg representations*, Commun. in Alg. 22 (1994), 1381-1393.

[28] B. Oliver, *p-stubborn subgroups of classical compact Lie groups*, J. Pure Appl. Algebra 92 (1994), 55-78.

[29] G. Segal, *Classifying spaces and spectral sequences*, Publ. Math. I.H.É.S. 34 (1968), 105-112

[30] G. Segal, *Cohomology of topological groups*, Symp Math IV (1970), 377-387.

[31] J.P. Serre, *Linear Representations of Finite Groups*, Springer-Verlag (1977).

[32] E.H. Spanier, *Algebraic topology*, McGraw Hill Book Company (1981).

[33] N. Steenrod, *The Topology of Fibre Bundles*, Prinston Math. Ser. 14 (1951).

[34] D. Sullivan, *Geometric topology, part I. Localization, periodicity and Galois symmetry*, mimeographed notes, M.I.T. (1970);

[35] R.M. Vogt, *Homotopy limits and colimits*, Math. Z. 134 (1973), 11-52.

[36] C.W. Wilkerson, *Self-maps of classifying spaces*, Localization in group theory and homotopy theory, Lecture Notes in Math. 418 (1974), 150-157.

[37] Z. Wojtkowiak, *On maps from* $\operatorname{hocolim} F$ *to* Z, Lecture Notes in Math., vol. 1298, Springer-Verlag, 1987, 227-236.

[38] A. Zabrodsky, *Maps between classifying spaces*, Annals of Math. Studies 113 (1987), 228-246.

[39] K. Ziemiański, *A faithful complex representation of the* 2*-compact group* $DI(4)$, Doctoral thesis, University of Warsaw (2005).

[40] K. Ziemiański, *A faithful unitary representation of the* 2*-compact group* $DI(4)$, J. Pure Appl. Algebra 213 (2009), 1239-1253.

[41] K. Ziemiański, *Homotopy representation of* $SO(7)$ *and* $Spin(7)$ *at prime* 2, J. Pure Appl. Algebra 212 (2008), 1525-1541.

In mathematics, specifically in homotopy theory, a classifying space BG of a topological group G is the quotient of a weakly contractible space EG (i.e. a topological space all of whose homotopy groups are trivial) by a proper free action of G. It has the property that any G principal bundle over a paracompact manifold is isomorphic to a pullback of the principal bundle EG â†' BG. As explained later, this means that classifying spaces represent a set-valued functor on the homotopy category of We can get an induced map Î±âˆ— : BG â†' BH by providing a natural transformation between the functors that these spaces repÂ resent: Î±âˆ— : {Gbundles over X} â†' {Hbundles over X}. Namely, send a Gbundle E â†' X to the Hbundle H Ã—G E â†' X where G acts on H through the homomorphism Î±. Thus B may be viewed. as a functor Topological groups â†' homotopy category of CWcomplexes. There is one particular case where the induced map admits a more explicit description. Suppose that H is a subLie group of a Lie group G. On a homework problem you showed that G â†' G/H was an Hbundle. It follows that the quotient EG â†' EG