

\mathcal{L} -INVARIANT OF p -ADIC L -FUNCTIONS

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Let $\overline{\mathbb{Q}} \subset \mathbb{C}$ be the field of all algebraic numbers. We fix a prime $p > 2$ and a p -adic absolute value $|\cdot|_p$ on $\overline{\mathbb{Q}}$. Then \mathbb{C}_p is the completion of $\overline{\mathbb{Q}}$ under $|\cdot|_p$. We write $W = \{x \in K \mid |x|_p < 1\}$ for the p -adic integer ring of sufficiently large extension K/\mathbb{Q}_p inside \mathbb{C}_p . We write $\overline{\mathbb{Q}}_p$ for the field of all numbers in \mathbb{C}_p algebraic over \mathbb{Q}_p .

We consider a Dirichlet series

$$\sum_{n=1}^{\infty} a_n n^{-s} = \prod_{\ell} \left(\sum_{n=0}^{\infty} a_{\ell^n} \ell^{-ns} \right) = \prod_{\ell} \left((1 - \alpha_{\ell}^{(1)} \ell^{-s})(1 - \alpha_{\ell}^{(2)} \ell^{-s}) \cdots (1 - \alpha_{\ell}^{(d)} \ell^{-s}) \right)^{-1}$$

with an Euler product over primes ℓ . Heuristically, if $|n^m - n^{m'}|_p < \varepsilon$ and $|a_n - b_n|_p < \varepsilon$ for all integers n , we would expect that

$$\left| \sum_{n=1}^{\infty} a_n n^m - \sum_{n=1}^{\infty} b_n n^m \right|_p < \varepsilon$$

up to some constant (the transcendental factor of the L -values) at good integers $s = m$. Even if $s \equiv s' \pmod{p^M(p-1)}$ for $g \gg 0$ ($\Leftrightarrow |s - s'|_p \leq p^{-M}$), p^s and $p^{s'}$ may not be very close p -adically, while $|\ell^s - \ell^{s'}|_p \leq p^{-1-M}$ if $\ell \neq p$. If m is negative, we will have further trouble interpolating the L -values if we do not remove Euler p -factors. Thus mod p class of L -values is better represented by the L -value with a certain **Euler p -factor removed**.

Here is an example. Start with a Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z}) \rightarrow \overline{\mathbb{Q}}^{\times}$ with $\chi(-1) = -1$. Here the word ‘‘Dirichlet character’’ means that it is multiplicative and $\chi(n) = 0$ if n has a nontrivial common factor with N . For positive integer m and m' , as long as $|n^m - n^{m'}|_p < 1$ (that is, $m \equiv m' \pmod{p-1}$), it is known from the time of Euler

$$|(1 - \chi(p)p^{m-1})L(1 - m, \chi) - (1 - \chi(p)p^{m'-1})L(1 - m', \chi)|_p < 1.$$

By a work of Kubota–Leopoldt and Iwasawa, we have a p -adic analytic L -function $L_{\chi}(s) = \Phi(\gamma^{1-s} - 1)$ for a power series $\Phi(X) \in \Lambda = W[[X]]$ and $\gamma = 1 + p$ such that

$$L_{\chi}(m) = \Phi(\gamma^{1-m} - 1) = (1 - \chi(p)p^{m-1})L(1 - m, \chi)$$

for all positive integer m as long as $|n^m - n|_p < 1$ for all n prime to p . The Iwasawa’s analyticity $L_{\chi}(s) = \Phi(\gamma^{1-s} - 1)$ guarantees that there are only finitely many zeros (counting with multiplicity) of $L_{\chi}(s)$ in W .

If we suppose $\chi = \left(\frac{-D}{\cdot}\right)$ for a square free positive integer D , the modifying Euler factor vanishes at $s = 1$ if the Legendre symbol $\left(\frac{-D}{p}\right) = 1 \Leftrightarrow (p) = \mathfrak{p}\overline{\mathfrak{p}}$ in $\mathbb{Z}[\sqrt{-D}]$

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with $\mathfrak{p} = \{x \in \mathbb{Z}[\sqrt{-D}] \mid |x|_p < 1\}$. In other words, $L_\chi(1) = 0$ and $\lambda \geq 1$. This type of zeros of a p -adic L -function is called an *exceptional zero*. We may regard χ as a Galois character $\text{Gal}(\mathbb{Q}[\mu_N]/\mathbb{Q}) = (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\chi} \{\pm 1\}$, and we remark that $\chi(\text{Frob}_p) = 1$ to have the exceptional zero. For a given p -adic L -function, we write e for the number of (linear) Euler factors producing the exceptional zero.

Here is another example. Start with an elliptic curve E/\mathbb{Q} defined by the equation $y^2 = 4x^3 - g_2x - g_3$ with $g_j \in \mathbb{Z}$. If $4x^3 - g_2x - g_3 \equiv 0 \pmod{\ell}$ has three distinct roots in $\overline{\mathbb{F}}_\ell$, the reduced curve $E_\ell = E \pmod{\ell}$ over \mathbb{F}_ℓ defined by $y^2 \equiv 4x^3 - g_2x - g_3 \pmod{\ell}$ remains to be an elliptic curve. Counting the number of points of $E_\ell(\mathbb{F}_\ell)$, we define $a_\ell = |\mathbf{P}^1(\mathbb{F}_\ell)| - |E_\ell(\mathbb{F}_\ell)| = 1 + \ell - |E_\ell(\mathbb{F}_\ell)|$. Then the Hasse-Weil L -function of E twisted by χ is given by

$$L(s, E, \chi) = \prod_{\ell} (1 - a_\ell \chi(\ell) \ell^{-s} + \chi(\ell)^2 \ell^{1-2s})^{-1} \quad (\text{Hasse}).$$

Take the Galois representation ρ_E of E on $\varprojlim_n \text{Ker}(p^n : E \rightarrow E) \cong \mathbb{Z}_p^2$. Then

$$L(s, E) = \prod_{\ell} \det(1 - \rho_E(\text{Frob}_\ell)|_{V_{I_\ell}} \ell^{-s})^{-1} \quad (\text{Weil}).$$

Split the Euler factor as a product of linear factors

$$(1 - a_p p^{-s} + p^{1-2s}) = (1 - \alpha p^{-s})(1 - \beta p^{-s}),$$

if one of α and β , say α , is a p -adic unit (so, $|\alpha|_p = 1$), E has either ordinary or multiplicative reduction modulo p . We suppose this *ordinarity condition*. Then by the solution of Shimura-Taniyama conjecture by Wiles et al, this L -function has p -adic analogue constructed by Mazur such that we have $\Phi_E(X) \in \Lambda$ with $\Phi_E(\varepsilon(\gamma) - 1) = (1 - \alpha^{-1} \varepsilon(p))^{\frac{G(\varepsilon^{-1})L(1, E, \varepsilon)}{\Omega_E}}$ for all p -power order character $\varepsilon : \mathbb{Z}_p^\times \rightarrow W^\times$; in other words, $L_p(s, E) = \Phi_E(\gamma^{1-s} - 1)$. Here Ω_E is the period of the Néron differential of E . The $\rho_E(\text{Frob}_p)$ has eigenvalue 1 if and only if E has multiplicative reduction mod p if and only if $E(\mathbb{C}_p) \cong \mathbb{C}_p^\times / q_E^{\mathbb{Z}}$ as Galois modules.

1. \mathcal{L} -INVARIANT

For a p -adic Galois representation ρ acting on $V \cong W^d$, we define $L(s, \rho) = \prod_{\ell} \det(1 - \rho(\text{Frob}_\ell)|_{V_{I_p}} p^{-s})^{-1}$, assuming that $\det(1 - \rho(\text{Frob}_\ell)|_{V_{I_\ell}} X) \in T[X]$ for a number field $T \subset \overline{\mathbb{Q}}$ independent of ℓ . We suppose that ρ is p -ordinary in the sense that ρ restricted to $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is upper triangular with diagonal characters \mathcal{N}^{a_j} on the inertia I_p for the p -adic cyclotomic character \mathcal{N} ordered from top to bottom as $a_1 \geq a_2 \geq \dots \geq 0 \geq \dots \geq a_d$. Thus

$$\rho|_{I_p} = \begin{pmatrix} \mathcal{N}^{a_1} & * & \dots & * \\ 0 & \mathcal{N}^{a_2} & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathcal{N}^{a_d} \end{pmatrix} \quad (\text{Ordinarity}).$$

Under the existence of a good analytic p -adic L -function $L_p(s, \rho)$ of Iwasawa type for ρ , we make

Conjecture 1.1. *If the eigenvalue of $Frob_p$ contains 1 with multiplicity e , then $L_p(s, \rho)$ has zero of order $e + \text{ord}_{s=1} L(s, \rho)$ and for a nonzero constant $\mathcal{L}(\rho) \in \mathbb{C}_p^\times$,*

$$\lim_{s \rightarrow 1} \frac{L_p(s, \rho)}{(s-1)^e} = \mathcal{L}(\rho) \mathcal{E}^+(\rho) \frac{L(1, \rho)}{c^+(\rho(1))},$$

where $c^+(\rho(1))$ is the transcendental factor of the critical complex L -value $L(1, \rho)$ and $\mathcal{E}^+(\rho)$ is the product of nonvanishing modifying p -factors.

The problem of \mathcal{L} -invariant is to compute explicitly the \mathcal{L} -invariant $\mathcal{L}(\rho)$. The \mathcal{L} -invariant in the cases where $\rho = \chi = \left(\frac{-D}{\cdot}\right)$ as above and $\rho = \rho_E$ for E with split multiplicative reduction is computed in the 1970s to 90s, and the results are

Theorem 1.2. *Let the notation and the assumption be as above.*

- (1) $\mathcal{L}(\chi) = \frac{\log_p(q)}{\text{ord}_p(q)} = \frac{\log_p(q)}{h}$ for $q \in \mathbb{C}_p$ given by $q = \varpi/\overline{\varpi}$, where h is the class number h of $\mathbb{Q}[\sqrt{-D}]$ and $\mathfrak{p}^h = (\varpi)$ (Gross–Koblitz and Ferrero–Greenberg);
- (2) For E split multiplicative at p , writing $E(\mathbb{C}_p) = \mathbb{C}_p^\times/q^{\mathbb{Z}}$ for the Tate period $q \in \mathbb{Q}_p^\times$, we have $\mathcal{L}(\rho_E) = \frac{\log_p(q)}{\text{ord}_p(q)}$. This was conjectured by Mazur–Tate–Teitelbaum and later proven by Greenberg–Stevens.

Here \log_p is the Iwasawa logarithm and $|x|_p = p^{-\text{ord}_p(x)}$.

Starting with a 2-dim p -adic Galois representation for a number field F , there is a systematic way to create many Galois representations whose eigenvalues of $Frob_p$ contain 1. Take a symmetric n -th tensor of ρ_E twisted by m times $\det(\rho)^{-1}$. Then $\rho_{n,m} = \rho_E^{\otimes n} \otimes \det(\rho_E)^{-m}$ has exceptional zero at $s = 1$ if $n = 2m$. If m is odd and F is totally real, $\rho_{2m,m}$ is critical at $s = 1$, and $e = |\{\mathfrak{p}|p\}|$.

There is an arithmetic way of constructing p -adic L -function due to Iwasawa and others. We can define Galois cohomologically the Selmer group

$$\text{Sel}_M(\rho) \subset H^1(\text{Gal}(\overline{\mathbb{Q}}/M), \rho \otimes \mathbb{Q}_p/\mathbb{Z}_p) \quad (M = F, F_\infty)$$

for the \mathbb{Z}_p -extension F_∞/F inside $F(\mu_{p^\infty})$. The Galois group $\Gamma = \text{Gal}(F_\infty/F)$ acts on $H^1(\text{Gal}(\overline{\mathbb{Q}}/F_\infty), \rho \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ and hence on $\text{Sel}_{F_\infty}(\rho)$, making it as a discrete module over the group algebra $W[[\Gamma]] = \varprojlim_n W[\Gamma/\Gamma^{p^n}]$. Identifying Γ with (a subgroup of) $1+p\mathbb{Z}_p$ by the cyclotomic character, we may regard $\gamma \in \Gamma$. Then $W[[\Gamma]] \cong \Lambda$ by $\gamma \mapsto 1+X$. By the classification theory of compact Λ -modules, the Pontryagin dual $\text{Sel}^*(\rho)$ is pseudo-isomorphic to $\prod_{f \in \Omega} \Lambda/f\Lambda$ for a finite set $\Omega \subset \Lambda$. The power series $\Phi_\rho = \prod_{f \in \Omega} f(X)$ is uniquely determined up to unit multiple. We then define $L_p^{\text{arith}}(s, \rho) = \Phi_\rho(\gamma^{1-s} - 1)$. Greenberg verified in 1994 the conjecture for this $L_p(s, \rho)$ except for the nonvanishing of $\mathcal{L}(\rho)$ (under some restrictive conditions). I also did verify in my book from Oxford university press his conjecture under milder assumptions by an automorphic way. If there exists a good analytic way of making the p -adic L -function $L_p^{\text{an}}(s, \rho) = \Phi_\rho^{\text{an}}(\gamma^{1-s} - 1)$ interpolating complex L -values, the main conjecture of Iwasawa's theory confirms $\Phi_\rho = \Phi_\rho^{\text{an}}$ up to unit multiple.

An important point is to describe \mathcal{L} without recourse to the above formula involving L -functions. Greenberg's computation of the \mathcal{L} -invariant is via Galois cohomology groups; for example, for $Ad(\rho) = \rho_{2,1}$: He found a unique subspace $\text{Sel}_F^{\text{cyc}}(\rho) \subset H^1(F, Ad(\rho))$ of dimension $e = |\{\mathfrak{p}|p\}|$ responsible to the order e zero at $s = 1$. This space is represented by cocycles $c_p : G_F = \text{Gal}(\overline{\mathbb{Q}}/F) \rightarrow Ad(\rho)$ such that

- (1) $c|_{D_{\mathfrak{p}}} \sim \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ for the decomposition group $D_{\mathfrak{p}}$ at each $\mathfrak{p}|p$;
- (2) c is unramified outside p and c modulo nilpotent matrices is unramified over $F_{\mathfrak{p}}[\mu_{p^\infty}]$ at all $\mathfrak{p}|p$ (automatic if $F_{\mathfrak{p}} = \mathbb{Q}_p$).

Take a basis $\{c_{\mathfrak{p}}\}_{\mathfrak{p}|p}$ of \mathcal{L} over K . Write

$$c_{\mathfrak{p}}(\sigma) \sim \begin{pmatrix} -a_{\mathfrak{p}}(\sigma) & * \\ 0 & a_{\mathfrak{p}}(\sigma) \end{pmatrix} \text{ for } \sigma \in D_{\mathfrak{p}'}$$

Then $a_{\mathfrak{p}} : D_{\mathfrak{p}'} \rightarrow K$ is a homomorphism. His \mathcal{L} -invariant is defined by

$$\mathcal{L}_{2,1} = \det \left((a_{\mathfrak{p}}([p, F_{\mathfrak{p}'}]_{\mathfrak{p}, \mathfrak{p}'}) (\log_p(\gamma) a_{\mathfrak{p}}([\gamma, F_{\mathfrak{p}'}]_{\mathfrak{p}, \mathfrak{p}'})^{-1}) \right).$$

The goal of this talk is to relate Greenberg's \mathcal{L} -invariant with Galois deformation theory, and give a couple of conjectures on $\mathcal{L}_{n,m}$ and the deformation ring.

2. A CONJECTURE

Take an elliptic curve E/F for a totally real field F with split multiplicative reduction modulo at every prime $\mathfrak{p}|p$; so, $E(\overline{F}_{\mathfrak{p}}) \cong \overline{F}_{\mathfrak{p}}^{\times} / q_{\mathfrak{p}}^{\mathbb{Z}}$ for $q_{\mathfrak{p}} \in F_{\mathfrak{p}}^{\times}$. Let $Q_{\mathfrak{p}} = N_{F_{\mathfrak{p}}/\mathbb{Q}_p}(q_{\mathfrak{p}})$.

Conjecture 2.1 (\mathcal{L} -invariant). *Suppose that the motive $Sym^{\otimes n}(H_1(E))(m)$ for $m \in \mathbb{Z}$ with $0 \leq m < n$ is critical at 1 (\Leftrightarrow either n is odd or $n = 2m$ with odd m). Then if the arithmetic or analytic $L_p(s, \rho_{n,m})$ has an exceptional zero at $s = 1$, we have*

$$\mathcal{L}_{n,m} = \prod_{\mathfrak{p}|p} \frac{\log_p(Q_{\mathfrak{p}})}{\text{ord}_p(Q_{\mathfrak{p}})}.$$

3. GALOIS DEFORMATION

The Greenberg's Selmer group $\text{Sel}_F^{cyc}(\rho)$ can be identified with the tangent space at the origin of the universal deformation space of $\rho_n = \rho_{n,0}$. Consider $J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and put $J_n = J_1^{\otimes n}$. We have ${}^t \rho_n(\sigma) J_n \rho_n(\sigma) = \mathcal{N}^n(\sigma) J_n$. Define an algebraic group G_n over \mathbb{Z}_p by

$$G_n(A) = \{ \alpha \in GL_{n+1}(A) \mid {}^t \alpha J_n \alpha = \nu(\alpha) J_n \}$$

for the similitude homomorphism $\nu : G_n \rightarrow \mathbb{G}_m$ (note that $G_1 = GL(2)$). The Galois representation ρ_n has values in $G_n(\mathbb{Z}_p)$. Consider the p -adic Lie algebra $Ad(\rho_n)$ of the derived group of G_n . Then $\sigma \in G_F$ acts on $Ad(\rho_n)$ by $X \mapsto \rho_n(\sigma) X \rho_n(\sigma)^{-1}$. Then

$$(3.1) \quad Ad(\rho_n) \cong \bigoplus_{j:\text{odd}, 1 \leq j \leq n} \rho_{2j,j}.$$

Start with ρ_n , and consider the deformation ring (R_n, ρ_n) which is universal among Galois representations: $\rho_A : G_F \rightarrow G_n(A) \equiv \rho_n \pmod{\mathfrak{m}_A}$ for local artinian \mathbb{Q}_p -algebras A with residue field \mathbb{Q}_p such that

- (Q_n1) unramified outside bad primes for E , ∞ and p ;
- (Q_n2) $\rho_A|_{D_{\mathfrak{p}}} \cong \begin{pmatrix} \alpha_{0,A,\mathfrak{p}} & * & \cdots & * \\ 0 & \alpha_{1,A,\mathfrak{p}} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_{n,A,\mathfrak{p}} \end{pmatrix}$ with $\alpha_{i,A,\mathfrak{p}} \equiv \mathcal{N}_{\mathfrak{p}}^{n-i} \pmod{\mathfrak{m}_A}$ with $\alpha_{i,A,\mathfrak{p}}|_{I_{\mathfrak{p}}}$ ($i = 0, 1, \dots, n$) factoring through the cyclotomic inertia group $\text{Gal}(F_{\mathfrak{p}}[\mu_{p^\infty}]/F_{\mathfrak{p}})$ for all $\mathfrak{p}|p$;
- (Q_n3) $\nu \circ \rho_A = \mathcal{N}^n$ for the global p -adic cyclotomic character \mathcal{N} .

The universal couple (R_n, ρ_n) exists under (Q_n1-3) . Each diagonal character, say, j -th δ_j from the top, of $\rho|_{D_p}$ induces a \mathbb{Z}_p -algebra homomorphism of $\mathbb{Z}_p[[X_{j,p}]]$ sending $1 + X_{j,p}$ to the value of δ_j at the generator of the p -primary part of $\text{Gal}(F_p[\mu_{p^\infty}]/F_p)$.

Conjecture 3.1. *We have $R_n \cong \mathbb{Q}_p[[X_{j,p}]]_{\mathfrak{p}|p}$, j :odd, $1 \leq j \leq n$ for variables $X_{j,p}$ induced by j -th diagonal character of $\rho_n|_{D_p}$; in particular, $\dim R_n = e \cdot \text{rank } G_n = e \lceil \frac{n}{2} \rceil$.*

When $F = \mathbb{Q}$ and $n = 1$, the conjecture holds, and for general totally real F , it holds if $n = 1$ and $\rho_E \pmod p$ has nonsoluble image (Wiles/Taylor, Skinner/Wiles, Fujiwara, Kisin, Khare/Wintenberger, Lin Chen). Since $G_3 \cong GSp(4)$ is the spin cover of $G_4 = GO(2, 3)$. Some progress has been made by A. Genestier and J. Tilouine towards the identification of Galois deformation rings and $GSp(4)$ -Hecke algebras (for $F = \mathbb{Q}$), there is a good prospect to get a proof of Conjecture 3.1 when $n = 3$ and 4. Further, when $F = \mathbb{Q}$, in view of the recent results of Clozel–Harris–Taylor and Taylor (in the paper proving the Sato–Tate conjecture for Tate curves), one would be able to treat general n in future not so far away. Conjecture 3.1 implies the \mathcal{L} -invariant conjecture for Greenberg’s \mathcal{L} -invariant:

Theorem 3.2. *Suppose Conjecture 3.1 and that n is odd. Then we have*

$$\prod_{j:\text{odd}, 0 < j \leq n} \mathcal{L}(\rho_{2j,j}) = \mathcal{L}(Ad(\rho_n)) = \prod_{\mathfrak{p}|p} \left(\frac{\log_p(Q_{\mathfrak{p}})}{\text{ord}_p(Q_{\mathfrak{p}})} \right)^{(n+1)/2}.$$

This follows from the fact that $\left(\frac{\partial \rho_n}{\partial X_{j,p}} \rho_n^{-1} \right) \Big|_{X=0}$ gives a canonical basis of $\text{Sel}^{cyc}(Ad(\rho_n))$; so, we can compute Greenberg’s \mathcal{L} -invariant explicitly.

For the abelian case, if $\chi = \left(\frac{M/F}{\mathfrak{p}} \right)$ for a CM field in which all $\mathfrak{p}|p$ in F splits into $\mathfrak{p}\overline{\mathfrak{p}}$ with $\mathfrak{p}^h = (\varpi(\mathfrak{p}))$, we also get

Corollary 3.3. *Up to a simple constant, for a half subset $\Sigma \sqcup \Sigma^c = \{\mathfrak{p}|p\}$, we have*

$$\mathcal{L}(\chi) = \frac{\det(\log_p(N_{\mathfrak{p}'}(\varpi(\mathfrak{P})^{(1-c)})))_{\mathfrak{p}, \mathfrak{p}' \in \Sigma}}{\prod_{\mathfrak{p} \in \Sigma} \text{ord}_p(N_{\mathfrak{p}}(\varpi(\mathfrak{P})^{(1-c)}))},$$

where $N_{\mathfrak{p}}$ is the local norm $N_{M_{\mathfrak{p}}/\mathbb{Q}_p}$ and c is a complex conjugation.

The above two results are obtained by explicitly computing the universal representation ρ . As for Corollary 3.3, we take a CM Hecke eigenform so that $\rho = \text{Ind}_M^F \psi$ for a CM Hecke character ψ of $\text{Gal}(\overline{\mathbb{Q}}/M)$. Writing κ for the universal character deforming ψ whose restriction to $\text{Gal}(\overline{\mathbb{Q}_p}/F_p)$ factors through $\text{Gal}(F[\mu_{p^\infty}]/F_p)$, we have $\kappa([u, M_{\mathfrak{p}}]) = (1 + X_{\mathfrak{p}})^{\log_p(N_{\mathfrak{p}}(u)/\log_p(\gamma))} \psi([u, M_{\mathfrak{p}}])$ for \mathfrak{p} -adic unit u , and we get $\rho = \text{Ind}_M^F \kappa$. By this fact, we can compute $\mathcal{L}(\text{Ind}_F^{\mathbb{Q}} \chi) = \mathcal{L}(\text{Ind}_F^{\mathbb{Q}} Ad(\rho))$.

The general case for all $n > 0$ is treated in my paper appeared in IMRN 2007 Vol. 2007, Article ID rnm102, 49 pages. doi:10.1093/imrn/rnm102, and the proof of the case: $n = 1$ is given in my book “Hilbert modular forms and Iwasawa theory” from Oxford University Press published in 2006.

Similarly, Kubota-Leopoldt's p-adic L-function can be constructed by interpolating Dirichlet L-functions at negative integers. My question is: how general is this method? For example, for which (L-functions of) automorphic representations have p-adic L-functions been constructed? I-functions p-adic-analysis nt.number-theory automorphic-forms. You should distinguish between the method (of which there is not one, but several) and the result (namely, the existence of p-adic L-functions). It is expected that p-adic L-function exist in great generality (say for algebraic automorphic representations, i.e. automorphic representations that are expected to have a connection with Galois representations). But they have not yet been constructed in that level of generality. Research Article. p-adic L functions and trivial zeroes. Bernadette Perrin-Riou. PNAS October 14, 1997 94 (21) 11118-11120; <https://doi.org/10.1073/pnas.94.21.11118>. We assume the existence of cpspec and the fact that the p-adic L function can be calculated by the formula $L_p(n) = \hat{\alpha}'(\text{cpspec}) \hat{\alpha}^{\wedge} n$. Theorem: The function L_p is nonzero at the trivial character 1 if and only if $\text{cpflach}(p) \hat{\alpha}^{\wedge} \mathbb{1}, \text{Hf1}(\hat{\alpha}, \mathbb{3}, T)$ and one has In particular, by using Flach's theorem (7), L_p is nonzero if and only if $\text{cpflach}(p)$ is nonzero. Assume $\text{cpflach}(p) \hat{\alpha} \neq 0$. Let $L_p, \text{sc} = L_p(e^{\wedge} 1 \hat{\alpha}^{\wedge} e^{\wedge} 2) \hat{\alpha}^{\wedge} \hat{\alpha}, \langle G \hat{\alpha}^{\wedge} \rangle$. Section 2. Greenberg Invariants. Section 3. p-adic L Functions. Section 4. Logarithm. Section 5. Special Systems and p-adic L Functions. Section 6. Some Theorems. Section 7. Even More Speculations.