# Review of *Philosophy of Mathematics* by Øystein Linnebo

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*Philosophy of Mathematics* by Øystein Linnebo is published in the *Princeton Foundations of Contemporary Philosophy* series, a series under the editorship of Scott Soames. It is a survey of some of the central questions, theories, and problems in the philosophy of mathematics. It is intended for graduate students and advanced undergraduate students in philosophy, as well as mathematicians and those interested in the foundations of mathematics. The book consists of twelve chapters, which together with the short introduction and back matter span 203 pages. Chapter one gives a very brief outline on the views of Plato and Kant on the philosophy of mathematics. This brief presentation of Kant's view, would have to be the most challenging part of the book for non-philosophers. Linnebo mentions in the introduction that space considerations led to some topics, including pre-Fregean philosophy of mathematics being given only the briefest treatment, and others not treated at all. Of the subjects not considered at all, Linnebo lists "Wittgenstein on mathematics, explanation in mathematical practice, the philosophy of mathematical practice, the use of experimental and nontraditional methods in mathematics, and new developments such as homotopy type theory."[2-3]

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After this short historical chapter, the next four deal with the three traditional positions in the philosophy of mathematics: Logicism (Frege's), formalism and intuitionism, with separate chapters on deductivist formalism and Hilbert's program. Chapter six deals with Mill's and Quine's empiricist views of mathematics. Chapter seven concerns nominalism and Hartry Field's attempts to nominalize science. Linnebo describes the first seven chapters as covering topics "that tend to be included in any good course in the philosophy of mathematics"[3]. The remaining five chapters Linnebo describes as more specialized and more philosophically and mathematically demanding. These chapters include mathematical intuition (discussing the views of Dagfinn Føllesdal, Gödel, Penelope Maddy, and Charles Parsons), neo-Fregean abstraction, the iterative conception of sets, structuralism, and the search for new axioms. Even these more specialized chapters deal with issues still central to the philosophy of mathematics.

Although the book is quite short, it uses its space quite efficiently. To illustrate this, I think it useful to go over what is contained in a fairly typical chapter. I will focus on chapter 11: Structuralism. This chapter is 16 pages long (neither particularly long or short compared with other chapters). The first section, of course, begins with a general discussion of structuralism. It then turns, in the second section, to eliminative structuralism. Here, as an example, the second order Dedekind-Peano axioms are presented and discussed, as well as Dedekind's *categoricity theorem*. After some advantages of the structuralist view of the natural numbers are outlined, the "catch" is mentioned that we need to assume the existence of at least one simply infinite system. Modal structuralism is discussed as a way of avoiding this catch, and there is a very brief discussion of some of the features of, and associated problems for, this view. The next section covers non-eliminative structuralism. This section briefly discusses views associated with Michael Resnik, Stewart Shapiro, and again Dedekind. The next section considers abstraction as a means of arriving at a structuralist view of mathematics. Here Linnebo sees some promise for a kind of Fregean abstaction giving us stuctures of the right kind. Two things occupy the same place in a structure if and only if there is an isomorphism mapping one to the other. The problem for this view is that, for instance, the "dumbbell graph":

$$\bullet \longleftrightarrow \bullet$$

has only one position instead of two (because each can be mapped isomorphically onto the other). Attempts to overcome this difficulty by considering structures as *structured universals* is then discussed. The final section of this chapter considers category theory's ability to capture structure. As an illustration of this, Linnebo considers the definition of the Cartesian product via a commutative diagram. He ends the chapter with a short discussion of the ontological assumptions of category theory. This is quite a lot to cover in a short chapter!

The book treats its material, and choice of material, from an explicitly opinionated perspective. On the first page Linnebo states: "I make no attempt to hide my own views concerning what is important and what works." This leads him to what he identifies as three themes of the book. The first of these themes is Fregeanism. That is not to say that Linnebo is always in agreement with Frege, but Frege does figure quite prominently in the book. Where Linnebo does want to agree with Frege is on the objectivity and independence of mathematical truths. The independence in question here is that mathematical propositions are counterfactually not dependent on the existence of humans or other rational life forms. As someone whose own views on the philosophy of mathematics are strongly influenced by Frege, I can find nothing to disagree with in terms of this aspect of the book.

The second theme is exploring attempts to avoid a *robust* or *full* platonistic understanding of mathematical objects. As we saw, Linnebo wants to defend the objectivity and mind-independence of mathematics, but he does not think this necessarily entails "full platonism." Linnebo admits that what this full platonism amounts to is not so clear: "a platonist goes further by making the (not very precise) claim that these objects are just as "real" as physical objects."[186] That said, Linnebo, discusses three ways one could be something of a platonist without being a robust or full platonist.[1] First, one could take mathematical objectivity as explanatorily prior to the existence of mathematical objects. Second, one could hold that mathematical objects exist only potentially. And third, one could hold a structuralist view of mathematical objects. Including the first of these as a way of avoiding robust realism, will rule out of the robust realist camp a lot of philosophers — perhaps too many. I have heard many philosophers of mathematics, with realist leanings, discuss the dictum, attributed to Kreisel, that "the problem is not the existence of mathematical objects but the objectivity of mathematical statements."[2] In each case the philosopher in question mentions this in agreement, and never, as far as I have seen, to disagree with it. As Linnebo notes, Frege is not to be counted as robust realist as he would have agreed with the order of priority expressed bu Kreisel's dictum. Gödel described mathematical intuition as beginning with reasoning about (formal) concepts and relations and arriving at objective knowledge of objects.[3] The fact that mathematical knowledge begins with reasoning about concepts and relations, would seem to be enough to rule Gödel out as a full platonist by Linnebo's criteria — despite his well known proclamation that intuition or mathematical reason is akin to sense experience. If both Frege and Gödel (perhaps the two most paradigmatic platonists) are excluded from being considered *full* platonists, one may wonder who would be included in this category.

The final theme that Linnebo identifies is the epistemology of mathematics. Linnebo describes his own position on the epistemology of mathematics as follows: " [m]y own orientation has been pluralist and gradualist. There appear to be several different sources of mathematical evidence, which gradually become less secure as they take us into the higher reaches of the subject."[187] The reasoning behind this position is presented most clearly in the chapter concerning the search for new axioms (chapter 12). We might have a variety of reasons for accepting a new mathematical axiom, and when we get into the higher reaches of set theory we may be getting further from anything that would count as strong evidence for an axiom. This is a reasonable position on the epistemology of mathematics. There is, however, one aspect of Linnebo's discussion of the epistemology of mathematics that I would like to discuss in more detail. In the introduction, Linnebo goes over the features that we standardly take mathematics to have and dubs the task of explaining how any science could have those features the *integration challenge*. He mentions Benacerraf (1973) as the version of this challenge most discussed in recent times, and says that he finds this unfortunate. The reason being that this presentation requires a causal

[1] See page 186 for the relevant discussion.

[2] This dictum and its attribution to Kreisel are discussed on page 32 of Linnebo's book.

[3] One can see talk of the priority of conceptual reasoning in, for instance, Gödel (1995a,b,c)

theory of reference. In chapter 7 where Linnebo discusses this in more detail, he writes: "[a]nd such an account, [Benacerraf] believes, has to be based on a causal connection between the agent in question and what is known." [102] Linnebo's presentation here is unfair to both Benacerraf and to commentators who discuss Benacerraf's classic paper. First, Benacerraf does not require that any epistemology for mathematics be based on a causal relation, but simply states his own preference for such theories:

> I favor a causal account of knowledge on which for *X* to know that *S* is true requires some causal relation to obtain between *X* and the referents of the names, predicates and quantifiers of *S*. I believe in addition in a causal theory of *reference*, thus making the link to my saying knowingly that *S doubly* causal. (Benacerraf, 1973, p. 671)

Beyond the odd-sounding talk of the referents of quantifiers, we see Benacerraf clearly describing the causal theory of knowledge and reference as being in accord with his philosophical *preference* and not as a requirement that any reasonable epistemology must meet. Benacerraf poses the challenge of trying to satisfy two competing desiderata for an epistemology of mathematics. These are, of course, that we would like, first, a straightforward semantics for mathematics, and, second, we would like for it to be clear how we know the mathematical truths we take ourselves to know. How well any account satisfies these desiderata (or horns of Benacerraf's dilemma as they have come to be called) is something we can ask of any pair consisting of a semantics and an epistemology for mathematics. The general challenge Benacerraf poses, is then independent of any particular semantics or epistemology. Benacerraf considers two principal examples of theories of mathematics truth (also discussed are Gödel and Plato's versions of platonism): one is *combinatorialism* which takes proof as the central semantic and epistemic notion, the other is the pairing of a straightforward Tarskian semantics with a causal theory of reference. Each of these fails to satisfy one of the desiderata, but these are just two possible answers to a very general challenge. The few views discussed in the paper are not meant to be exhaustive of the space of possibilities. If the paper were merely an argument to the effect that platonism is incompatible with a causal theory of knowledge, then Linnebo would be correct that it is unfortunate that it is the focus of so much contemporary discussion.[4] We can ask of any philosophical theory of mathematical truth (which includes both a semantics and an epistemology) how well it satisfies each of Benacerraf's desiderata, and this is why Benacerraf's paper continues to draw so much attention to this day.

Also against Benacerraf's position, Linnebo says "Moreover, in a clash between a philosophical theory of knowledge and the successful science of mathematics, it seems foolhardy to side with the former. Mathematics commands

[4] In fairness, the tie to the causal theory of knowledge is just one reason Linnebo thinks the focus on Benacerraf's paper is unfortunate. The other is that the tension that Benacerraf discusses already figure in, Plato's and Kant's philosophy of mathematics. However, even if Benacerraf's paper was not the first to deal with this tension, it is at least a clear and vivid portrayal of the tension.

far greater confidence than any philosophical analysis of knowledge." [104] But what is at issue here is whether we should abandon mathematical *platonism*, a philosophical theory about mathematics, for epistemological reasons, not whether we should abandon mathematics because it conflicts with one's epistemological principles. Benacerraf, after all, is explicit that he takes it as a given that we do actually know the (vast majority of) mathematical truths which we take ourselves to know.

All in all, this book covers in impressive amount of material in a very efficient manner. It does so with great clarity and so will be useful for its intended audience. It is, as mentioned, explicitly an opinionated in a way that shapes both the choice of material discussed as well as the evaluation of the views discussed. The material chosen is a wide variety of topics central to the philosophy of mathematics, even if, obviously, not exhaustive. The evaluations of the views Linnebo discusses, even if guided by his own opinions, tends to be reasonable and fair. Considerations of space usually prevent him from doing more than mentioning the advantages and drawbacks of each position, before pointing the reader to further material (as is done at the end of each chapter). As such this book provides a nice *lay of the land* for anyone interested in contemporary philosophy of mathematics.

## References


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by Ã˜ystein Linnebo. This book is an important contribution to the philosophy of mathematics. It aims to clarify and answer questions about realism in connection with mathematics, in particular whether there exist mathematical objects (ontological realism) more. This book is an important contribution to the philosophy of mathematics. It aims to clarify and answer questions about realism in connection with mathematics, in particular whether there exist mathematical objects (ontological realism) and whether all meaningful mathematical Publisher: Oxford Univ Press. Ã˜ystein Linnebo (born 1971) is a Norwegian philosopher. As of 2020 he is currently employed in the Department of Philosophy at the University of Oslo, having earlier held a position as Professor of Philosophy at Birkbeck College, University of London. He is a fellow of the Norwegian Academy of Science and Letters. Linnebo earned his MA in Mathematics from the University of Oslo in 1995 and his PhD in Philosophy at Harvard University in June 2002.